MR 3 : sequences of functions

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Pointwise convergence

Definition (Pointwise convergence)

Let $E \subset \mathbb{R}$ and (f_n) a sequence of functions defined on E. (f_n) converges pointwisely to f defined on E if :
$\forall x \in E$, the numerical sequence $(f_n(x))$ converges to $f(x)$.

Example

Prove that the sequence defined by $f_n : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & 1 + \frac{x}{n} \end{array} \right.$ converges pointwisely to a certain function.

Supremum

- Let A be a subset of $\mathbb{R}$. The supremum of A, denoted $\sup A$, is the **smallest upper bound** of A. If A has no upper bound, then its supremum is $+\infty$.
- Let $f : E \to \mathbb{R}$ be a bounded function. We denote by $\|f\|_\infty$ the supremum of the set $\{|f(x)| \mid x \in \mathbb{E}\}$.

Be careful ! $\|f\|_\infty$ does not exist if f is not bounded.

Uniform convergence

Definition (Uniform convergence)

Let $E \subset \mathbb{R}$ and (f_n) a sequence of functions defined on E. (f_n) converges uniformly to f defined on E if :
$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \forall x \in E, |f_n(x) - f(x)| < \epsilon.$

Or, equivalently :

$(f_n - f)$ is bounded starting from a certain rank, and the numerical sequence $(\|f_n - f\|_\infty)$ converges to 0.

Example

Prove that the sequence defined by $f_n : \left| \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \exp(-x + \frac{1}{n}) \end{array} \right.$

converges uniformly to $f : \left| \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \exp(-x) \end{array} \right.$.

Links between the two

- Prove that if (f_n) converges uniformly to f on $E \subset \mathbb{R}$, then (f_n) converges pointwisely to f on $E \subset \mathbb{R}$.
- However, the converse is not true : prove it using the following sequence as a counter example :

$$f_n : \left| \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \exp(\frac{-x}{n}) \end{array} \right.$$

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Other convergence modes

Definition (L^1 convergence)

Let I be an interval of $\mathbb{R}$, (f_n) be a sequence of integrables functions defined on I, and f be an integrable function defined on I. (f_n) converges to f in L^1 sense if :

$$\int_I |f_n(x) - f(x)|dx \xrightarrow{n\to+\infty} 0$$

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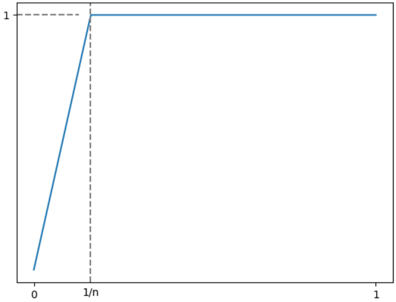
L^1 convergence does not imply pointwise convergence

Graphe de f_n :



- Write f_n as a function of x.
- Prove that f_n converges to $\mathbb{1}_{[0,1]}$ in L^1 sense, but that it does not converges pointwisely to this function.

Uniform convergence and continuous functions

Reminder (Continuity)

Let $f : \mathbb{R} \to \mathbb{R}$ *be a function and* $z \in \mathbb{R}$. *Write the definition of the continuity of* f *at* z.

Exercise

Let I *be an interval of* $\mathbb{R}$, *and* (f_n) *a sequence of continuous functions on* I. *Prove that if* (f_n) *converges uniformly to* f, *then* f *is also continuous.*

Normal convergence

Definition (Normal convergence)

Let I be an interval of $\mathbb{R}$, and (f_n) a sequence of bounded functions. The function series $\sum f_n$ is said to converge normally if the numerical series $\sum \|f_n\|_\infty$ is convergent.

Theorem

Let I be an interval of $\mathbb{R}$, and (f_n) a sequence of bounded functions. If the function series $\sum f_n$ converges **normally** *to S, then it also converges* **uniformly** *to S.*

Exercise

- Prove that the function $S : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \sum_{n=1}^{+\infty} \frac{\sin(nx)}{n^2} \end{array} \right.$ is well defined.
- Prove that the function series that defines S converges normally.
- Deduce that S is continuous.

Thank you!