MR 6 : Lebesgue integral

Centralesupélec

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Markov inequality

Exercise

Let $(E, \mathcal{F}, \mu)$ *be a measure space and* $f : E \to \mathbb{R}_+$ *a measureable function. Prove that for any* $\alpha > 0$*, the Markov inequality holds :*

$$\alpha\mu(\{x \in E \mid f(x) > \alpha\}) \leq \int_E f \, d\mu.$$

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- Hint : notice that the term $\alpha\mu(\{x \in E \mid f(x) > \alpha\})$ can be written as the integral of a certain function.

A little kahoot quizz !

Pushforward measure

Definition (Pushforward measure)

Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be 2 measureable spaces. Let $f : E \to F$ be a measurable function, and ν be a measure on the σ-algebra $\mathcal{E}$. We define the pushforward of ν :

$$\forall B \in \mathcal{F},\ \nu_f(B) = \nu(f^{-1}(B)).$$

Prove that ν_f defines a measure on the σ-algebra $\mathcal{F}$.

To prove that ν_f is a measure on$\mathcal{F}$, you have to prove that :

- ν_f is a function from $\mathcal{F}$ to $\mathbb{R}_+$.
- $\nu_f(\emptyset) = 0$.
- For any pairwise disjoint collection of sets (B_n) of $\mathcal{F}$, we have :

$$\nu_f(\bigcup_{n=0}^{\infty} B_n) = \sum_{n=0}^{\infty} \nu_f(B_n).$$

Integral with respect to the pushforward measure

Let $\phi : F \to \mathbb{R}_+$ be a measurable function. Recall the construction of the integral :

$$\int_F \phi(x)\, \nu_f(dx).$$

Result

Deduce the following equality :

$$\int_F \phi(x)\,\nu_f(dx) = \int_E \phi(f(x))\,\nu(dx).$$

Hint :

- Prove it for $\phi = \mathbb{1}_B$ with $B \in \mathcal{F}$.
- For any positive measurable function f, there exists a non decreasing sequence of positive simple functions (f_n) that converges pointwise to f (see exercise 3.5 of your booklet).