MR PDE : Sobolev spaces

Centralesupélec

January 9, 2025

Poincaré inequality

Theorem

There exists a constant $C > 0$ *which only depends on* $b - a$ *such that for all* $u \in H_0^1(a, b)$, $\|u\|_{L^2(a,b)} \leq C\|u'\|_{L^2(a,b)}$.

Poincaré inequality

Theorem

There exists a constant $C > 0$ which only depends on $b - a$ such that for all $u \in H_0^1(a, b)$, $\|u\|_{L^2(a,b)} \leq C\|u'\|_{L^2(a,b)}$.

- **BE CAREFUL :** this inequality is only true on $H_0^1(a, b)$.

Poincaré inequality

Theorem

There exists a constant $C > 0$ which only depends on $b - a$ such that for all $u \in H_0^1(a,b)$, $\|u\|_{L^2(a,b)} \leq C\|u'\|_{L^2(a,b)}$.

- **BE CAREFUL :** this inequality is only true on $H_0^1(a,b)$.
- This is why $\|u\|_{H_0^1(a,b)} := \|u'\|_{L^2(a,b)}$ defines a norm on $H_0^1(a,b)$.

An exercise

Exercise

We define $\phi : \left| \begin{array}{ccc} H^1(0,1) & \longrightarrow & \mathbb{R} \\ u & \longmapsto & u(0) \end{array} \right.$ *. Prove that* ϕ *is a well defined continuous linear form.*

An exercise

Exercise

We define $\phi : \left| \begin{array}{ccc} H^1(0,1) & \longrightarrow & \mathbb{R} \\ u & \longmapsto & u(0) \end{array} \right.$ *. Prove that* ϕ *is a well defined continuous linear form.*

Hint

Notice that there exists $C > 0$ *such that for all* $u \in H^1(0,1)$*, it holds :*
$\forall x \in [0,1], |u(x)| \leq C\|u\|_{H^1(0,1)}$.

An exercise

Exercise

We define $\psi : \left| \begin{array}{ccc} H_0^1(0,1) & \longrightarrow & \mathbb{R} \\ u & \longmapsto & u(0) \end{array} \right.$. *Prove that* ψ *is a well defined continuous linear form.*

The Dirac distribution

Reminder

The Dirac distribution is defined on $\mathcal{D}(-1,1)$ *by* $\langle \delta, \phi \rangle = \phi(0)$.

Exercise

Prove that $\delta \notin L^2(-1,1)$. *To do so, suppose that there exists* $f \in L^2(-1,1)$ *such that, for all* $\phi \in \mathcal{D}(-1,1)$, *it holds :*

$$\langle \delta, \phi \rangle = \int_{-1}^{1} f\phi \, d\lambda.$$