Projet Fluide S8

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Introduction

We are interested in the space $H(\mathbf{curl},\Omega)$, where Ω is a domain in $\mathbb{R}^N$ with $N=2$ or $N=3$, for which we assume an extension property similar to that of H^1 extension domains. The objective is to develop a trace theory on $\mathbf{H}(\mathbf{curl},\Omega)$, which will allow us to generalize existing results on Lipschitz domains (see [10]), with giving the definitions of various boundary operators such as the tangential trace or the normal trace. For this purpose, we will introduce a capacity, the definition of which will be adapted to $\mathbf{H}(\mathbf{curl},\Omega)$, and prove the existence of quasi continuous representatives as it has already been done to build the trace operator in $\mathrm{H}^1(\Omega)$ in [2]. This will allow us to give meaning to new types of boundary conditions, as it has been done in [6]. We also give the first steps to generalizing the Hodge theory on simply connected domains.

Notations

Every function or space of functions written in bold is vector. Definitions of the objects below will be given later in the article.

$\|\cdot\|$	is the euclidean norm on $\mathbb{R}^N$.
$\|\cdot\|_E$	is the norm on the vector space E.
Ω	is a domain in $\mathbb{R}^N$.
$\mathcal{Z}u$	is the extension by zero of u, a function defined on Ω, outside of Ω.
$\mathrm{H}^1(\Omega)$	is the classical Sobolev space of elements of $\mathrm{L}^2(\Omega)$ which have a gradient in $\mathbf{L}^2(\Omega)$.
Cap	is the H^1 capacity.
$\mathrm{B}_1(\partial\Omega)$	is the space of all q.e. (for the H^1 capacity) equivalence classes of pointwise restriction $w\big\|_{\partial\Omega}$ of quasi continuous representatives of functions in $\mathrm{H}^1(\Omega)$.
$\mathbf{H}(\mathbf{curl},\Omega)$	is the classical Sobolev space of elements in $\mathbf{L}^2(\Omega)$ which have a divergence in $\mathrm{L}^2(\Omega)$.
$\mathbf{H}(\mathbf{curl},\Omega)$	is the classical Sobolev space of elements in $\mathbf{L}^2(\Omega)$ which have a **curl** in $\mathbf{L}^2(\Omega)$.
$\mathrm{Cap}_{\mathbf{curl}}$	is the H^1 capacity.
$\mathbf{B}_{\mathrm{curl}}(\partial\Omega)$	is the space of all q.e. (for the $\mathbf{H}$(curl) capacity) equivalence classes of pointwise restrictions $w\big\|_{\partial\Omega}$ of quasi-continuous representatives of functions in $\mathbf{H}(\mathrm{curl},\Omega)$.
Tr	is the trace operator on $\mathrm{H}^1(\Omega)$ to $\mathrm{B}_1(\partial\Omega)$.
$\mathbf{Tr}_{\mathbf{curl}}$	is the trace operator on $\mathrm{H}^1(\Omega)$ to $\mathbf{B}_{\mathbf{curl}}(\partial\Omega)$.

1 Definitions

In this whole report, when we say domain, we mean an open **simply connected** subset of the vector space we're considering.

Definition 1.1. *We define for* Ω *a domain in* $\mathbb{R}^N$ *the space* $\mathbf{H}(\mathbf{curl},\Omega) = \{\mathbf{w} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl}\,\mathbf{w} \in \mathbf{L}^2(\Omega)\}$*. This is a Hilbert space with the following norm:*

$$\|\mathbf{u}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} := \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\,\mathbf{curl}(\mathbf{u})\|^2_{\mathbf{L}^2(\Omega)}.$$

Definition 1.2. *We define for* Ω *a domain in* $\mathbb{R}^N$ *the space* $\mathbf{H}_0(\mathbf{curl},\Omega) := \overline{\mathcal{D}(\Omega)^N}^{\|\cdot\|_{\mathbf{H}(\mathbf{curl},\Omega)}}$*. This is a closed subspace of* $\mathbf{H}(\mathbf{curl},\Omega)$*.*

Definition 1.3. *We define for* Ω *a domain in* $\mathbb{R}^N$ *the space* $\mathbf{H}(\mathrm{div},\Omega) = \{\mathbf{w} \in \mathbf{L}^2(\Omega) \mid \mathrm{div}\,\mathbf{w} \in \mathrm{L}^2(\Omega)\}$*. This is a Hilbert space with the following norm:*

$$\|\mathbf{u}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} := \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\,\mathrm{div}(\mathbf{u})\|^2_{\mathrm{L}^2(\Omega)}.$$

Definition 1.4. *We define for* Ω *a domain in* $\mathbb{R}^N$ *the space* $\mathbf{H}_0(\mathrm{div},\Omega) := \overline{\mathcal{D}(\Omega)^N}^{\|\cdot\|_{\mathbf{H}(\mathrm{div},\Omega)}}$*. This is a closed subspace of* $\mathbf{H}(\mathrm{div},\Omega)$*.*

Definition 1.5 (Property ($\mathcal{P}1$))**.**

We say that Ω *satisfies property* ($\mathcal{P}1$) *if there exists a continuous operator:*

$$E : \mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{H}(\mathrm{curl},\mathbb{R}^N).$$

Such that for all $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl},\Omega)$*, we have* $E\boldsymbol{u}\big|_{\Omega} = \boldsymbol{u}$*. And the boundary* $\partial\Omega$ *has strictly positive* $\mathbf{H}(\mathbf{curl})$ *capacity.*

This definition is directly inspired by the more classical definition of H^1 extension domains, which are well characterized. We also observe that if Ω satisfies ($\mathcal{P}1$), then we also have a continuous extension operator $\mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{H}(\mathrm{curl},\Theta)$ for any domain Θ containing Ω, by restricting the extension to $\mathbb{R}^N$ to Θ.

Definition 1.6 (Sobolev admissible domains)**.** *As defined in [8], we say that* Ω *is a Sobolev admissible domain if it is a Sobolev* H^1*-extension domain and its boundary* $\partial\Omega$ *has strictly positive* H^1 *capacity (see [8]) for the definition of* H^1 *capacity).*

Definition 1.7 (Property ($\mathcal{P}2$))**.**

We say that Ω *satisfies property* ($\mathcal{P}2$) *if it satisfies* ($\mathcal{P}1$) *and is a Sobolev admissible domain as well.*

Definition 1.8. *We define* $\mathcal{D}(\overline{\Omega}) := \{u|_{\Omega} \mid u \in \mathcal{D}(\mathbb{R}^N)\}$*.*

One of the main interests in working with domains satisfying ($\mathcal{P}1$) is the following result.

Theorem 1.1. *Let* Ω *be a domain in* $\mathbb{R}^N$ *satisfying property* ($\mathcal{P}1$)*. Then* $\mathcal{D}(\overline{\Omega})^N$ *is dense in* $\mathbf{H}(\mathbf{curl},\Omega)$*.*

Proof. Let $\boldsymbol{f} \in \mathbf{H}(\mathbf{curl},\Omega)$. It is known from [10] that $\mathcal{D}(\mathbb{R}^N)^N$ is dense in $\mathbf{H}(\mathrm{curl},\mathbb{R}^N)$. Therefore, there exists a sequence $(\boldsymbol{g}_n) \subset \mathcal{D}(\mathbb{R}^N)^N$ which converges to $E\boldsymbol{f}$ in $\mathbf{H}(\mathrm{curl},\mathbb{R}^N)$. Now we have:

$$\begin{aligned}
\|E\boldsymbol{f} - \boldsymbol{g}_n\|^2_{\mathbf{H}(\mathrm{curl},\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \|E\boldsymbol{f} - \boldsymbol{g}_n\|^2\,dx + \int_{\mathbb{R}^N} \|\,\mathbf{curl}(E\boldsymbol{f} - \boldsymbol{g}_n)\|^2\,dx \\
&= \int_{\Omega} \|\boldsymbol{f} - \boldsymbol{g}_n\|^2\,dx + \int_{\Omega} \|\,\mathbf{curl}(\boldsymbol{f} - \boldsymbol{g}_n)\|^2\,dx + \|E\boldsymbol{f} - \boldsymbol{g}_n\|^2_{\mathbf{H}(\mathrm{curl},\mathbb{R}^N\setminus\Omega)} \\
&= \|\boldsymbol{f} - \boldsymbol{g}_n\big|_{\Omega}\|^2_{\mathbf{H}(\mathrm{curl},\Omega)} + \|E\boldsymbol{f} - \boldsymbol{g}_n\|^2_{\mathbf{H}(\mathrm{curl},\mathbb{R}^N\setminus\Omega)}.
\end{aligned}$$

Since $E\boldsymbol{f}$ equals $\boldsymbol{f}$ on Ω, we get $\|\boldsymbol{f} - \boldsymbol{g}_n\big|_{\Omega}\|^2_{\mathbf{H}(\mathrm{curl},\Omega)} \leq \|E\boldsymbol{f} - \boldsymbol{g}_n\|^2_{\mathbf{H}(\mathrm{curl},\mathbb{R}^N)}$, and we deduce that $\boldsymbol{g}_n\big|_{\Omega} \xrightarrow[n\to\infty]{} \boldsymbol{f}$. Moreover, for all n, $\boldsymbol{g}_n \in \mathcal{D}(\overline{\Omega})^N$. □

2 Normal trace on $\mathbf{H}(\mathrm{div},\Omega)$

Lemma 2.1. *Let Ω be a Sobolev admissible domain of $\mathbb{R}^N$, of boundary Γ, and $u \in H^1(\Omega)$. We denote by $\mathcal{Z}u$ the extension by zero of u oustide of Ω. If $\mathcal{Z}u \in H^1(\mathbb{R}^N)$, then $u \in H^1_0(\Omega)$.*

Proof. To do so we will use proposition 2.4 of [2], and we will denote by Cap the H^1-capacity that is defined in this book. Let $x \in \mathbb{R}^N \setminus \overline{\Omega}$. Then for r small enough we have $B(x,r) \cap \Omega = \emptyset$, so $\mathcal{Z}u$ is zero on $B(x,r)$. We deduce that $\tilde{\mathcal{Z}u}(x) = 0$ for all $x \in \mathbb{R}^N \setminus \overline{\Omega}$.

Suppose there exists $A \subset \mathbb{R}^N \setminus \Omega$ with strictly positive capacity such that $\tilde{\mathcal{Z}u}$ is nonzero on all of A. We have seen that $\tilde{\mathcal{Z}u}$ is zero on $\mathbb{R}^N \setminus \overline{\Omega}$, so we must have $A \subset \Gamma$. Since $\tilde{\mathcal{Z}u}$ is $H^1(\mathbb{R}^N)$-quasi-continuous, there exists an open set G of $\mathbb{R}^N$ such that $Cap(G) < \frac{Cap(A)}{2}$ and $\tilde{\mathcal{Z}u}$ is continuous on $\mathbb{R}^N \setminus G$. Because the capacity is an outer measure, we deduce that there exists $x \in A$ such that $x \notin G$. We have $x \in \Gamma$, hence $x \in \overline{\mathbb{R}^N \setminus \overline{\Omega}}$. So there exists a sequence $(x_n) \subset (\mathbb{R}^N \setminus \overline{\Omega})$ that converges to x. On the other hand, G is open, and $x \notin G$, so from some rank n_0 onwards the terms of (x_n) are in $(\mathbb{R}^N \setminus \overline{\Omega}) \cap (\mathbb{R}^N \setminus G)$. But we have for all $n \geq n_0$ the equality $\tilde{\mathcal{Z}u}(x_n) = 0$ since $x_n \in \mathbb{R}^N \setminus \overline{\Omega}$, so $\tilde{\mathcal{Z}u}(x_n) \to 0$. But this contradicts the sequential characterization of the continuity of $\tilde{\mathcal{Z}u}$ on $(\mathbb{R}^N \setminus G)$, since by hypothesis we had $x \in A$ hence $\tilde{\mathcal{Z}u}(x) \neq 0$.

We thus know that $\tilde{\mathcal{Z}u}$ is zero $H^1(\mathbb{R}^N)$-quasi-everywhere on $\mathbb{R}^N \setminus \Omega$. From proposition 2.1 of [2], this implies that $(\mathcal{Z}u)|_\Omega = u \in H^1_0(\Omega)$. ☐

Theorem 2.1. *Let Ω be a Sobolev admissible domain of $\mathbb{R}^N$. Then $\mathcal{D}(\overline{\Omega})^N$ is dense in $\mathbf{H}(\mathrm{div},\Omega)$.*

Proof. Let $f \in \mathbf{H}(\mathrm{div},\Omega)'$. We can apply the Riesz theorem and fix $l \in H(\mathrm{div},\Omega)$ such that:

$$\forall u \in H(\mathrm{div},\Omega), \langle f,u\rangle = \sum_{i=1}^{N} \langle l_i, u_i\rangle_{L^2(\Omega)} + \langle l_{N+1}, \mathrm{div}(u)\rangle_{L^2(\Omega)}$$

with $l_{N+1} = \mathrm{div}(l)$. Suppose that f vanishes on $\mathcal{D}(\bar{\Omega})^N$ and denote by $\mathcal{Z}l_i$ the extension by zero of l_i to $\mathbb{R}^N$. We deduce:

$$\forall \phi \in \mathcal{D}(\mathbb{R}^N)^N, \langle \mathcal{Z}l, \phi\rangle_{H(\mathrm{div},\mathbb{R}^N)} = \int_{\mathbb{R}^N} (\mathcal{Z}l \cdot \phi + \mathcal{Z}l_{N+1} \,\mathrm{div}\, \phi)\, dx = 0.$$

This last equation is equivalent to saying that in $(\mathcal{D}(\mathbb{R}^N)^N)'$, we have $\mathcal{Z}l = \nabla(\mathcal{Z}l_{N+1})$. According to the du Bois-Reymond lemma, we deduce that $\mathcal{Z}l = \nabla(\mathcal{Z}l_{N+1})$ in $L^2(\mathbb{R}^N)^N$. Hence $\mathcal{Z}l_{N+1} \in H^1(\mathbb{R}^N)$: according to lemma 2.1, we deduce that $l_{N+1} \in H^1_0(\Omega)$. We can therefore find a sequence $(\psi_\mu) \subset \mathcal{D}(\Omega)$ that converges to l_{N+1} in $H^1(\Omega)$. By continuity of the operator ∇ on $H^1_0(\Omega)$, we know that the sequence $(\nabla(\psi_\mu))$ converges to $\nabla(l_{N+1}) = l$ in $H^1(\Omega)$. Let $u \in H(\mathrm{div};\Omega)$; by definition of the distribution $\mathrm{div}(u)$, we have:

$$\forall \mu \in \mathbb{N}, \int_\Omega \nabla(\psi_\mu) \cdot u \, dx + \int_\Omega \psi_\mu \,\mathrm{div}(u)\, dx = 0.$$

We pass to the limit by continuity of the L^2 inner product in $H^1(\Omega)$, and we obtain $\langle f,u\rangle_{H(\mathrm{div};\Omega)} = 0$. We deduce that f vanishes on the whole space $H(\mathrm{div};\Omega)$, and thus that $\mathcal{D}(\overline{\Omega})^N$ is dense in $\mathbf{H}(\mathrm{div},\Omega)$. ☐

Now we may define the normal trace operator as it has already been done in [7].

Definition 2.1. *Let Ω be a Sobolev admissible domain. Then there exists a normal trace operator:*

$$\mathrm{Tr}_n \left| \begin{array}{ccc} \mathbf{H}(\mathrm{div},\Omega) & \longrightarrow & \mathbf{B}'_1(\partial\Omega) \\ \boldsymbol{u} & \longmapsto & \mathrm{Tr}_n(\boldsymbol{u}) \end{array} \right.$$

which is linear, continuous and verifies for all $v \in \mathrm{H}^1(\Omega)$*:*

$$\langle \mathrm{Tr}_n(\boldsymbol{u}), \mathrm{Tr}(v) \rangle_{\mathrm{B_1}'(\Omega), \mathrm{B_1}(\Omega)} = \int_\Omega \mathrm{div}(\boldsymbol{u}) v \, dx + \int_\Omega \boldsymbol{u} \cdot \nabla v \, dx. \tag{1}$$

Proof. See theorem 3.1 of [7]. □

Theorem 2.2. *Let* Ω *be a Sobolev admissible domain in* $\mathbb{R}^N$*. Then it holds:* $\mathrm{Ker}(\mathrm{Tr}_n) = \mathbf{H}_0(\mathrm{div}, \Omega)$.

Proof. See proposition 3.4 of [7]. □

3 Trace theory on $\mathbf{H}(\mathbf{curl}, \Omega)$

This section is based on the similar theory developed for H^1 in [2]. Nevertheless, the entire section using existing results on capacities and Dirichlet forms should be carefully reviewed to ensure that working with vector-valued functions rather than scalar ones does not cause any issues. A priori, this should not change anything, as it amounts to working with scalar functions in $L^2(\Omega \times \{1, \dots, n\})$.

Definition 3.1 ($\mathbf{H}(\mathbf{curl})$ - capacity)**.**

Let $A \subset \mathbb{R}^N$*. We define the* $\mathbf{H}(\mathbf{curl})$ *- capacity of* A *as:*

$\mathrm{Cap}_{\mathbf{curl}}(A) := \inf\{\|\mathbf{u}\|_{\mathbf{H}(\mathrm{curl}, \mathbb{R}^N)} : \mathbf{u} \in \mathbf{H}(\mathrm{curl}, \mathbb{R}^N)$ *s.t.* $\exists \mathcal{O} \subset \mathbb{R}^N$ *open s.t.* $A \subset \mathcal{O}$ *and* $\mathbf{u} \geq 1$ *a.e. on* $\mathcal{O}\}$.

Where $\mathbf{u} \geq 1$ *means that each component of* $\mathbf{u}$ *is* ≥ 1.

Definition 3.2 ($\mathbf{H}(\mathbf{curl}, \Omega)$ - relative capacity)**.**

Let Ω *be a domain satisfying* $(\mathcal{P}1)$*. Let* $A \subset \overline{\Omega}$*. We define the* $\mathbf{H}(\mathbf{curl})$ *- relative capacity of* A *as:*

$$\begin{aligned}\mathrm{Cap}^{\overline{\Omega}}_{\mathbf{curl}}(A) := \inf\{\|\mathbf{u}\|_{\mathbf{H}(\mathrm{curl}, \mathbb{R}^N)} : \mathbf{u} \in \mathbf{H}(\mathrm{curl}, \mathbb{R}^N) \ \text{s.t.}\ \exists \mathcal{O} \subset \mathbb{R}^N \ \text{open s.t.} \\ A \subset \mathcal{O} \ \text{and}\ \mathbf{u} \geq 1 \ \text{a.e. on}\ \mathcal{O} \cap \Omega\}.\end{aligned}$$

Where $\mathbf{u} \geq 1$ *means that each component of* $\mathbf{u}$ *is* ≥ 1.

Proposition 3.1. *Let* Ω *satisfy* $(\mathcal{P}1)$*. Then for all* $A \subset \overline{\Omega}$*, we have:* $\mathrm{Cap}^{\overline{\Omega}}_{\mathbf{curl}}(A) \leq \mathrm{Cap}_{\mathbf{curl}}(A)$.

Note that if $\Omega = \mathbb{R}^N$, then these two definitions coincide.
As in [2], we observe that this definition of the relative capacity corresponds to a particular case of capacity associated with a Dirichlet form (see [4]). Let Ω be a domain satisfying $(\mathcal{P}1)$. We set $X = \overline{\Omega}$, $\mathcal{B}(X)$ the set of Borel subsets of X, and m the measure on $\mathcal{B}(X)$ defined by $m(E) = \lambda^{(N)}(E \cap \Omega)$. Then we have $\mathbf{L}^2(\Omega) = \mathbf{L}^2(X, \mathcal{B}(X), m)$, and we consider the Dirichlet form $(\mathcal{E}, \mathbb{D})$ with $\mathbb{D} = \mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathcal{E}(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{curl}(\mathbf{u}) \cdot \mathbf{curl}(\mathbf{v}) \, dx$.

Property (T) from [4, p. 52] is satisfied since $\mathbf{H}(\mathbf{curl}, \Omega)$ contains $\mathcal{D}(\Omega)^3$, which is dense in $L^2(\Omega)$ by Lemma 1.1 of [10]. Moreover, according to Theorem 1.1 proven in the previous section, property (D) from [4, p. 54] is also satisfied. The relative capacity is then exactly the capacity associated with $(\mathcal{E}, \mathbb{D})$ in the sense of [4, def. 8.1.1, p. 52]. This allows us to directly deduce the following results:

Lemma 3.1. *Let* Ω *be a domain satisfying* $(\mathcal{P}1)$*. Then we have the following properties:*

1. *The relative capacity on* Ω *is an outer measure.*
2. *For every Borel set* $A \subset \overline{\Omega}$*, we have* $\lambda^{(N)}(A) \leq \mathrm{Cap}^{\overline{\Omega}}_{\mathbf{curl}}(A)$.

A set is said to be polar (resp. relatively polar) if it has zero (resp. relative) capacity. A property is said to hold quasi-everywhere or q.e. (resp. relatively quasi-everywhere or r.q.e.) if it holds outside a polar (resp. relatively polar) set.

Proof. These are results on capacities associated with a Dirichlet form, as shown in [4]. ☐

Definition 3.3 (Quasi-continuity)**.**

We say that $u : \mathbb{R}^N \to \mathbb{R}$ *is quasi-continuous if:*

$$\forall \epsilon > 0,\ \exists G \text{ an open set in } \mathbb{R}^N,\ \mathrm{Cap}_{\mathbf{curl}}(G) < \epsilon,\ \text{and } u \text{ is continuous on } \mathbb{R}^N \setminus G.$$

Definition 3.4 (Relative quasi-continuity)**.**

Let Ω *satisfy* $(\mathcal{P}1)$*. We say that* $u : \overline{\Omega} \to \mathbb{R}$ *is relatively quasi-continuous if:*

$$\forall \epsilon > 0,\ \exists G \text{ an open set in } \mathbb{R}^N,\ \mathrm{Cap}^{\overline{\Omega}}_{\mathbf{curl}}(G \cap \overline{\Omega}) < \epsilon,\ \text{and } u \text{ is continuous on } \mathbb{R}^N \setminus G.$$

Proposition 3.2. *Let* Ω *satisfy* $(\mathcal{P}1)$ *and* f *be a quasi-continuous function; then* $f\big|_{\overline{\Omega}}$ *is relatively quasi-continuous.*

Proof. This follows from Proposition 3.1. ☐

Proposition 3.3. *Let* Ω *satisfy* $(\mathcal{P}1)$*, and* $f : \overline{\Omega} \to \mathbb{R}$ *be a relatively quasi-continuous function. Let* $V \subset \overline{\Omega}$ *be an open set such that* $f \geq 0$ *m-almost everywhere on* V*. Then* $f \geq 0$ *relatively almost everywhere on* V*.*

Proof. See [4, prop. 8.1.6, p. 54]. ☐

We have already justified that the relative capacity satisfies properties $(D), (T)$ from [4]. This leads to the following very important result.

Theorem 3.1. *Let* Ω *satisfy* $(\mathcal{P}1)$*. Then every* $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ *admits a representative* $\tilde{\mathbf{u}} : \overline{\Omega} \to \mathbb{R}^3$ *which is relatively quasi-continuous, unique up to r.q.e. equality.*

Proof. First, let $E : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}(\mathrm{curl}, \mathbb{R}^N)$ be an extension operator, which exists because Ω verifies $(\mathcal{P}1)$. By [4, prop. 8.2.1, p. 54], we know that there exists a quasi-continuous representative $\tilde{E\boldsymbol{u}}$ of $E\boldsymbol{u}$. By proposition 3.2, we know that $\tilde{E\boldsymbol{u}}\big|_{\overline{\Omega}}$ is relatively quasi continous on $\overline{\Omega}$; this is a generic result on Dirichlet forms with domain that is dense in L^2.
For the uniqueness, let us consider E_1 and E_2 two extension operators. Then by the definition of extension operators, we have $\tilde{E_1\boldsymbol{u}}\big|_{\overline{\Omega}} = \boldsymbol{u} = \tilde{E_2\boldsymbol{u}}\big|_{\overline{\Omega}}$ $\lambda^{(N)}$ a.e. on Ω. Because $\tilde{E_j\boldsymbol{u}}\big|_{\overline{\Omega}}$ is relatively quasi continuous on $\overline{\Omega}$, it can be deduced from proposition 3.3 that $\tilde{E_1\boldsymbol{u}}\big|_{\overline{\Omega}} = \tilde{E_2\boldsymbol{u}}\big|_{\overline{\Omega}}$ r.q.e. on $\overline{\Omega}$. ☐

Definition 3.5. *Let* $\mathbf{u} \in \mathbf{H}(\mathrm{curl}, \mathbb{R}^N)$*; we say that an open set* G *is a* 1*-regular set of* $\mathbf{u}$ *if for every* $\mathbf{v} \in \mathbf{H}(\mathrm{curl}, \mathbb{R}^N) \cap \mathbf{C}_0(\mathbb{R}^N)$ *with support in* G*, it holds :* $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{H}(\mathrm{curl}, \mathbb{R}^N)} = 0$.

It is proven in [9, page 79] that the union of two 1-regular sets for some $\mathbf{u}$ is still 1-regular. This enables us to define the 1-spectrum set of $\mathbf{u}$.

Definition 3.6. *Let* $\mathbf{u} \in \mathbf{H}(\mathrm{curl}, \mathbb{R}^N)$*. We define the* 1*-spectrum of* $\mathbf{u}$*,* $\sigma_1(\mathbf{u})$ *as the complement of the largest* 1*-regular open set of* $\mathbf{u}$*.*

Definition 3.7. *Let* $F \subset \mathbb{R}^N$*; we define the space* $\mathbf{W}_1^F$ *as the closure in* $\mathbf{H}(\mathrm{curl}, \mathbb{R}^N)$ *of the space of functions* $\mathbf{u} \in \mathbf{H}(\mathrm{curl}, \mathbb{R}^N)$ *with* $\sigma_1(\mathbf{u})$*.*

Theorem 3.2. *Let* Ω *a domain verifying* $(\mathcal{P}1)$*. Then we have the following:*

$$\mathbf{H}_0(\mathbf{curl}, \Omega) = \{\mathbf{u}\big|_{\Omega} : \ \mathbf{u} \in \mathbf{H}(\mathrm{curl}, \mathbb{R}^N) \ s.t\ \tilde{\mathbf{u}} = 0 \ q.e. \ sur\ \mathbb{R}^N \setminus \Omega\}.$$

Proof. We may regard $\mathbf{H}_0(\mathbf{curl},\Omega)$ as a closed subspace of $\mathbf{H}(\mathrm{curl},\mathbb{R}^N)$, by extending the functions outside of Ω by 0. Then it can proved (see [9, Problem 3.3.4, p. 81]) that $\mathbf{H}_0(\mathbf{curl},\Omega)$ is the orthogonal complement of $\mathbf{W}_1^{\mathbb{R}^N\setminus\Omega}$ in $\mathbf{H}(\mathrm{curl},\mathbb{R}^N)$. Besides, by [9, theorem 3.3.4, p. 80], we know that :

$$\{\mathbf{u} : \ \mathbf{u}\in\mathbf{H}(\mathrm{curl},\mathbb{R}^N) \text{ s.t } \tilde{\mathbf{u}}=0 \text{ q.e. sur } \mathbb{R}^N\setminus\Omega\} = (\mathbf{W}_1^{\mathbb{R}^N\setminus\Omega})^{\perp}.$$

As all of these are closed subspaces of $\mathbf{H}(\mathrm{curl},\mathbb{R}^N)$, we finally obtain :

$$\{\mathbf{u}\big|_{\Omega} : \ \mathbf{u}\in\mathbf{H}(\mathrm{curl},\mathbb{R}^N) \text{ s.t } \tilde{\mathbf{u}}=0 \text{ q.e. sur } \mathbb{R}^N\setminus\Omega\} = \mathbf{H}_0(\mathbf{curl},\Omega).$$

$\square$

Theorem 3.3. *Let Ω be a domain verifying $(\mathcal{P}1)$. Then we have:*

$$\mathbf{H}_0(\mathbf{curl},\Omega) = \{\mathbf{u}\in\mathbf{H}(\mathbf{curl},\Omega) \ s.t \ \tilde{\mathbf{u}}=0 \ on \ \partial\Omega\}.$$

Proof. This can be directly deduced from theorem 3.2 and proposition 3.3. $\square$

Lemma 3.2. *Let Ω be a domain verifying $(\mathcal{P}1)$. The following decomposition holds :*

$$\mathbf{H}(\mathbf{curl},\Omega) = \mathbf{H}_0(\mathbf{curl},\Omega)\oplus\mathbf{H}_0(\mathbf{curl},\Omega)^{\perp}.$$

And the space $\mathbf{H}_0(\mathbf{curl},\Omega)^{\perp}$ can be characterized as the set of functions in $\mathbf{H}(\mathrm{curl},\Omega)$ that verify :

$$\mathbf{curl}(\mathbf{curl}(\mathbf{u}))+\mathbf{u}=0 \ dans \ \mathcal{D}'(\Omega)^N. \tag{2}$$

Proof. Since $\mathbf{H}_0(\mathbf{curl},\Omega)$ is a closed subspace, we immediately obtain the orthogonal decomposition. As for the characterization of $\mathbf{H}_0(\mathbf{curl},\Omega)^{\perp}$, let $\mathbf{u}\in\mathbf{H}_0(\mathbf{curl},\Omega)^{\perp}$. In particular, $\mathbf{u}$ is orthogonal in $\mathbf{H}(\mathbf{curl},\Omega)$ to every $\boldsymbol{\phi}\in\mathcal{D}(\Omega)^N$:

$$\begin{aligned} \langle\mathbf{u},\boldsymbol{\phi}\rangle_{\mathbf{H}(\mathbf{curl},\Omega)} &= 0\\ \int_\Omega \mathbf{u}\cdot\boldsymbol{\phi}\,dx + \int_\Omega \mathbf{curl}(\mathbf{u})\cdot\mathbf{curl}(\boldsymbol{\phi})\,dx &= 0\\ [\mathbf{u},\boldsymbol{\phi}] + [\mathbf{curl}(\mathbf{curl}(\mathbf{u})),\boldsymbol{\phi}] &= 0. \end{aligned}$$

Where $[\cdot,\cdot]$ denotes the duality bracket between $\mathcal{D}'(\Omega)^N$ and $\mathcal{D}(\Omega)^N$. Thus, $\mathbf{u}$ indeed satisfies 2. Conversely, if $\mathbf{u}$ satisfies 2, by reversing the previous computation, we obtain that $\mathbf{u}$ is orthogonal to every $\boldsymbol{\phi}\in\mathcal{D}(\Omega)^N$. By continuity of the scalar product, it then follows that $\mathbf{u}$ is orthogonal to all of $\mathbf{H}_0(\mathbf{curl},\Omega)$. $\square$

Definition 3.8 (Trace operator on $\mathbf{H}(\mathbf{curl},\Omega)$)**.**

Let $\boldsymbol{u}\in\mathbf{H}(\mathbf{curl},\Omega)$; according to Theorem 3.1, one can find a quasi-continuous representative $\tilde{\boldsymbol{u}}$ of $\boldsymbol{u}$. We then define the trace of $\boldsymbol{u}$ as:

$$\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{u}) = \tilde{\boldsymbol{u}}\big|_{\partial\Omega}.$$

The trace is thus defined in a quasi-unique way on $\partial\Omega$. We denote by $\mathbf{B}_{\mathbf{curl}}(\partial\Omega)$ the image of $\mathbf{H}(\mathrm{curl},\Omega)$ by $\mathbf{Tr}_{\mathbf{curl}}$.

Theorem 3.4. *We equip the space $\mathbf{B}_{\mathbf{curl}}(\partial\Omega)$ with the following norm:*

$$\|\boldsymbol{e}\|_{\mathbf{B}_{\mathbf{curl}}(\partial\Omega)} := \inf\{\|\boldsymbol{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)}, \ \boldsymbol{u}\in\mathbf{H}(\mathbf{curl},\Omega), \ s.t. \ \mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{u})=\boldsymbol{e} \ q.e.\}.$$

This norm makes $\mathbf{B}_{\mathbf{curl}}(\partial\Omega)$ into a Hilbert space.

Proof. Let $\boldsymbol{e}\in\mathbf{B}_{\mathbf{curl}}(\partial\Omega)$. According to Lemma 3.2, the following problem is well-posed in $\mathbf{H}(\mathbf{curl},\Omega)$:

$$\begin{cases} \mathbf{curl}(\mathbf{curl}(\boldsymbol{u}))+\boldsymbol{u}=0\\ \mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{u})=\boldsymbol{e} \end{cases}$$

Let $\boldsymbol{u}_{\boldsymbol{e}}$ be the unique solution. By Lemma 3.2, we have $\boldsymbol{u}_{\boldsymbol{e}}\in\mathbf{H}_0(\mathbf{curl},\Omega)^{\perp}$. Any $\boldsymbol{u}\in\mathbf{H}(\mathbf{curl},\Omega)$

satisfying $\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{u}) = \boldsymbol{e}$ can thus be written as $\boldsymbol{u} = \boldsymbol{u_e} + \boldsymbol{u_0}$ with $\boldsymbol{u}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$. Therefore:

$$\inf\{\|\boldsymbol{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)},\ \boldsymbol{u} \in \mathbf{H}(\mathbf{curl}, \Omega),\ \text{s.t. } \mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{u}) = \boldsymbol{e} \text{ q.e.}\} = \inf\{\|\boldsymbol{u_e}+\boldsymbol{u_0}\|_{\mathbf{H}(\mathbf{curl},\Omega)},\ \boldsymbol{u}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)\}.$$

Now by the Pythagorean theorem, for any $\boldsymbol{u}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$, we have $\|\boldsymbol{u_e} + \boldsymbol{u}_0\|_{\mathbf{H}(\mathbf{curl},\Omega)} \geq \|\boldsymbol{u_e}\|_{\mathbf{H}(\mathbf{curl},\Omega)}$. Hence the inf is attained at $\boldsymbol{u_e}$.

The norm of $\boldsymbol{e}$ can therefore be written as $\|\boldsymbol{e}\|_{\mathbf{B}_{\mathbf{curl}}(\partial\Omega)} = \|\boldsymbol{u_e}\|_{\mathbf{H}(\mathbf{curl},\Omega)}$. This indeed defines a norm on $\mathbf{B}_{\mathbf{curl}}(\partial\Omega)$, and it can easily be checked that it makes $\mathbf{B}_{\mathbf{curl}}(\partial\Omega)$ a Hilbert space, since $\mathbf{H}(\mathbf{curl}, \Omega)$ endowed with $\|\cdot\|_{\mathbf{H}(\mathbf{curl},\Omega)}$ is a Hilbert space. □

4 Tangential trace

Lemma 4.1. *Let Ω satisfy $(\mathcal{P}1)$. A function $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ belongs to $\mathbf{H}_0(\mathbf{curl}, \Omega)$ if and only if:*

$$\int_\Omega \boldsymbol{u} \cdot \mathbf{curl}(\boldsymbol{\phi}), dx - \int_\Omega \mathbf{curl}(\boldsymbol{u}) \cdot \boldsymbol{\phi}, dx = 0. \tag{3}$$

For all $\boldsymbol{\phi} \in \mathcal{D}(\overline{\Omega})^N$.

Proof. We first discard the case where $\Omega = \mathbb{R}^N$, since we then have $\mathbf{H}(\text{curl}, R^N) = \mathbf{H}_0(\mathbf{curl}, \mathbb{R}^N)$ (see [10]). We therefore assume from now on that $\Omega \subsetneq \mathbb{R}^N$.

Direct implication:

We know that equation (3) is always satisfied when $\mathbf{u} \in \mathcal{D}(\Omega)^N$ (definition of the distributional **curl**). Now, every element of $\mathbf{H}_0(\mathbf{curl}, \Omega)$ is the limit of elements in $\mathcal{D}(\Omega)^N$, so we know that (3) holds for every element of $\mathbf{H}_0(\mathbf{curl}, \Omega)$.

Converse implication:

Let $\mathbf{u}$ satisfy (3). We define $\mathcal{Z}\mathbf{u}$ as the extension by 0 of $\mathbf{u}$ to $\mathbb{R}^N$. Equation (3) allows us to justify that $\mathcal{Z}\mathbf{u} \in \mathbf{H}(\text{curl}, \mathbb{R}^N)$. By Theorem 3.1, let $\tilde{\mathcal{Z}\mathbf{u}}$ be the quasi-continuous representative of $\mathcal{Z}\mathbf{u}$. We have $\tilde{\mathcal{Z}\mathbf{u}} = 0$ $\lambda^{(N)}$-almost everywhere on $\mathbb{R}^N \setminus \overline{\Omega}$, and thus, by Proposition 3.2, we get $\tilde{\mathcal{Z}\mathbf{u}} = 0$ quasi-everywhere on $\mathbb{R}^N \setminus \overline{\Omega}$ since this set is open. We denote by V the set of zero capacity such that $\tilde{\mathcal{Z}\mathbf{u}} = 0$ on $\mathbb{R}^N \setminus V$.

Suppose there exists $A \subset \Omega$ of strictly positive capacity such that $\tilde{\mathcal{Z}\mathbf{u}} \neq 0$ on A. Since we have seen that this function is quasi-continuous on $\mathbb{R}^N \setminus \overline{\Omega}$, it must be that $A \subset \partial\Omega$. By the quasi-continuity of $\tilde{\mathcal{Z}\mathbf{u}}$, one can find an open set G in R^N such that $\tilde{\mathcal{Z}\mathbf{u}}$ is continuous on $R^N \setminus G$ and $\text{Cap}_{\mathbf{curl}}(G) < \frac{\text{Cap}_{\mathbf{curl}}(A)}{2}$. Since capacity is an outer measure, this implies the existence of $x \in A$ such that $x \notin G$. We have $x \in \partial\Omega$, so we can find a sequence (x_n) with terms in $\mathbb{R}^N \setminus \overline{\Omega}$ converging to x. Moreover, we can ensure that each term of (x_n) lies in $\mathbb{R}^N \setminus V$. Indeed, since $\text{Cap}_{\mathbf{curl}}(V) = 0$, by property 2 of Lemma 3.1, we also have $\lambda^{(N)}(V) = 0$. In particular, V has empty interior.

We thus have a sequence (x_n) such that $\tilde{\mathcal{Z}\mathbf{u}}(x_n) = 0$ for all n and $x_n \to x$. Furthermore, since G is open and $x \notin G$, from some point onward, the terms of (x_n) all lie outside G. Since $\tilde{\mathcal{Z}\mathbf{u}}$ is continuous on $\mathbb{R}^N \setminus G$, we should have $\tilde{\mathcal{Z}\mathbf{u}}(x_n) \to \tilde{\mathcal{Z}\mathbf{u}}(x)$, hence $\tilde{\mathcal{Z}\mathbf{u}}(x) = 0$, which contradicts the fact that $x \in A$.

We thus conclude that $\tilde{\mathcal{Z}\mathbf{u}} = 0$ quasi-everywhere on $\mathbb{R}^N \setminus \Omega$. By Theorem 3.2, we therefore arrive at the conclusion that $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$, since $\mathbf{u} = \mathcal{Z}\mathbf{u}\big|_\Omega$. □

At this point we can define the tangential trace operator.

Theorem 4.1 (Tangential Trace)**.**

Let Ω a domain satisfying $(\mathcal{P}1)$. There exists a bounded tangential trace operator $\mathbf{Tr}_{\mathrm{T}} : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{B}'_{\mathbf{curl}}(\partial\Omega)$, and the following identity holds for all $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$:

$$\langle \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}), \mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{v}) \rangle_{\mathbf{B}'_{\mathbf{curl}}(\partial\Omega), \mathbf{B}_{\mathbf{curl}}(\partial\Omega)} = \int_{\Omega} \boldsymbol{u} \cdot \mathbf{curl}(\boldsymbol{v})\, dx - \int_{\Omega} \boldsymbol{v} \cdot \mathbf{curl}(\boldsymbol{u})\, dx. \tag{4}$$

Proof. Since $\mathbf{Tr}_{\mathbf{curl}}\big|_{\mathbf{H}_0(\mathbf{curl},\Omega)^{\perp}} : \mathbf{H}_0(\mathbf{curl}, \Omega)^{\perp} \to \mathbf{B}_{\mathbf{curl}}(\partial\Omega)$ is an isomorphism, it admits a bounded inverse $\boldsymbol{\theta} : \mathbf{B}'_{\mathbf{curl}}(\partial\Omega) \to \mathbf{H}_0(\mathbf{curl}, \Omega)^{\perp}$. The following identity holds :

$$\forall \boldsymbol{v} \in \mathbf{H}(\mathbf{curl}, \Omega),\ \boldsymbol{\theta}(\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{v})) = \boldsymbol{v}^{\perp}. \tag{5}$$

Where $\boldsymbol{v}^{\perp}$ denotes the projection of $\boldsymbol{v}$ onto $\mathbf{H}_0(\mathbf{curl}, \Omega)^{\perp}$.

Let $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$. We define the element $\mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u})$ of $\mathbf{B}'_{\mathbf{curl}}(\partial\Omega)$ by the following duality relation, for all $\boldsymbol{e} \in \mathbf{B}_{\mathbf{curl}}(\partial\Omega)$:

$$\langle \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}), \boldsymbol{e} \rangle_{\mathbf{B}'_{\mathbf{curl}}(\Omega), \mathbf{B}_{\mathbf{curl}}(\Omega)} = \int_{\Omega} \mathbf{curl}(\boldsymbol{\theta}(\boldsymbol{e})) \cdot \boldsymbol{u}\, dx - \int_{\Omega} \mathbf{curl}(\boldsymbol{u}) \cdot \boldsymbol{\theta}(\boldsymbol{e})\, dx.$$

By composition, this clearly defines a continuous linear form on $\mathbf{B}_{\mathbf{curl}}(\partial\Omega)$. We now verify indentity(4) ; Let $\boldsymbol{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$. We obtain, from (5) :

$$\langle \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}), \mathbf{Tr}(\boldsymbol{v}) \rangle_{\mathbf{B}'_{\mathbf{curl}}(\partial\Omega), \mathbf{B}_{\mathbf{curl}}(\partial\Omega)} = \int_{\Omega} \mathbf{curl}(\boldsymbol{v}^{\perp}) \cdot \boldsymbol{u}\, dx - \int_{\Omega} \mathbf{curl}(\boldsymbol{u}) \cdot \boldsymbol{v}^{\perp}\, dx. \tag{6}$$

Now write $\boldsymbol{v} = \boldsymbol{v}^{\perp} + \boldsymbol{v}_0$, where $\boldsymbol{v}_0$ is the projection of $\boldsymbol{v}$ onto $\mathbf{H}_0(\mathbf{curl}, \Omega)$. And according to lemma 4.1, we have $\int_{\Omega} \mathbf{curl}(\boldsymbol{u}) \cdot \boldsymbol{v}_0\, dx - \int_{\Omega} \mathbf{curl}(\boldsymbol{v}_0) \cdot \boldsymbol{u}\, dx = 0$, which allows us to conclude identity (4).

It remains to show that the tangential trace operator defined above is continuous. Let $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{e} \in \mathbf{B}_{\mathbf{curl}}(\partial\Omega)$ with norm 1. Using Cauchy-Schwarz it leads to :

$$|\langle \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}), \mathbf{Tr}(\boldsymbol{v}) \rangle_{\mathbf{B}'_{\mathbf{curl}}(\partial\Omega), \mathbf{B}_{\mathbf{curl}}(\partial\Omega)}| \leq \| \mathbf{curl}(\boldsymbol{u}) \|_{\mathbf{L}^2(\Omega)} \|\boldsymbol{\theta}(\boldsymbol{e})\|_{\mathbf{L}^2(\Omega)} + \| \mathbf{curl}(\boldsymbol{\theta}(\boldsymbol{e})) \|_{\mathbf{L}^2(\Omega)} \|\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)}.$$

Since $\boldsymbol{\theta}$ is continuous from $\mathbf{B}_{\mathbf{curl}}(\partial\Omega) \to \mathbf{H}(\mathbf{curl}, \Omega)$, there exists constants $C_1 > 0$ and $C_2 > 0$ such that :

$$\begin{aligned} \|\boldsymbol{\theta}(\boldsymbol{e})\|_{\mathbf{L}^2(\Omega)} &\leq C_1 \|\boldsymbol{e}\|_{\mathbf{B}_{\mathbf{curl}}(\partial\Omega)} \\ \| \mathbf{curl}(\boldsymbol{\theta}(\boldsymbol{e})) \|_{\mathbf{L}^2(\Omega)} &\leq C_2 \|\boldsymbol{e}\|_{\mathbf{B}_{\mathbf{curl}}(\partial\Omega)}. \end{aligned}$$

Setting $C = \max(C_1, C_2)$ we obtain :

$$\begin{aligned} |\langle \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}), \mathbf{Tr}(\boldsymbol{v}) \rangle_{\mathbf{B}'_{\mathbf{curl}}(\partial\Omega), \mathbf{B}_{\mathbf{curl}}(\partial\Omega)}| &\leq C \underbrace{\|\boldsymbol{e}\|_{\mathbf{B}_{\mathbf{curl}}(\partial\Omega)}}_{=1} (\| \mathbf{curl}(\boldsymbol{u}) \|_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)}) \\ &\leq C' \|\boldsymbol{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)}. \end{aligned}$$

Since this holds for all $\boldsymbol{e}$ with norm 1, we deduce $\| \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}) \|_{\mathbf{B}'_{\mathbf{curl}}(\partial\Omega)} \leq C' \|\boldsymbol{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)}$. This proves the continuity of $\mathbf{Tr}_{\mathrm{T}}$. □

Proposition 4.1. *Let Ω be a domain verifying $(\mathcal{P}1)$. Then the kernel of the operator $\mathbf{Tr}_{\mathrm{T}}$ is exactly $\mathbf{H}_0(\mathbf{curl}, \Omega)$.*

Proof. This is simply another way to state lemma 4.1. □

Definition 4.1. *Let Ω be a domain verifying $(\mathcal{P}1)$. We define the normal part of $\mathbf{B}_{\mathbf{curl}}(\Omega)$ the following way :*

$$\mathbf{B}_{\perp}(\partial\Omega) = \{\boldsymbol{e} \in \mathbf{B}_{\mathbf{curl}}(\partial\Omega),\ \forall \boldsymbol{v} \in \mathbf{H}(\mathbf{curl}, \Omega),\ \langle \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{v}), \boldsymbol{e} \rangle_{\mathbf{B}'_{\mathbf{curl}}(\partial\Omega), \mathbf{B}_{\mathbf{curl}}(\partial\Omega)} = 0\}. \tag{7}$$

Lemma 4.2. *Let Ω be a domain verifying $(\mathcal{P}1)$. The following decomposition of $\mathbf{H}(\mathbf{curl},\Omega)$ holds :*

$$\mathbf{B}_{\mathbf{curl}}(\partial\Omega) = \mathbf{B}_{\perp}(\partial\Omega) \oplus \mathbf{B}_T(\partial\Omega). \tag{8}$$

Where $\mathbf{B}_T(\partial\Omega)$, the tangential part $\mathbf{B}_1(\partial\Omega)$, is the orthogonal complementary of $\mathbf{B}_{\perp}(\partial\Omega)$.

Proof. It is sufficient to prove that $\mathbf{B}_{\perp}(\partial\Omega)$ is closed in $\mathbf{B}_{\mathbf{curl}}(\partial\Omega)$. And we know that :

$$\mathbf{B}_{\perp}(\partial\Omega) = \bigcap_{\boldsymbol{v}\in\mathbf{H}(\mathbf{curl},\Omega)} \mathrm{Ker}(\mathbf{Tr}_{\mathrm{T}}(\mathrm{v}))\,.$$

We have justified that $\mathbf{Tr}_{\mathrm{T}}(\boldsymbol{v})$ is continous on $\mathbf{B}_{\mathbf{curl}}(\partial\Omega)$ for all $\boldsymbol{v} \in \mathbf{H}(\mathbf{curl},\Omega)$, so each kernel is closed. As an intersection of closed sets, $\mathbf{B}_{\perp}(\partial\Omega)$ is hence closed in $\mathbf{B}_{\mathbf{curl}}(\partial\Omega)$. □

Definition 4.2. *Let $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl},\Omega)$. We call the projection of $\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{u})$ on $\mathbf{B}_T(\partial\Omega)$, noted $(\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{u}))_T$ the tangential component of $\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{u})$. Therefore,*

$$\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{u}) = (\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{u}))_T + (\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{u}))_{\perp}.$$

Given the formula (4) and the definition of the space $\mathbf{B}_{\perp}(\partial\Omega)$, we notice that for every $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl},\Omega)$, $\mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u})$ might be seen as an element of $\mathbf{B}'_T(\partial\Omega)$. The purpose of this new perspective is that the operator $\mathbf{Tr}_{\mathrm{T}}$ is onto from $\mathbf{H}(\mathbf{curl},\Omega)$ to $\mathbf{B}'_T(\partial\Omega)$, which was not the case before.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^N$ verifying $(\mathcal{P}1)$. Then for every $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl},\Omega)$ and $\boldsymbol{v} \in \mathbf{H}(\mathbf{curl},\Omega)$ the following holds :*

$$\langle \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}), (\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{v}))_T \rangle_{\mathbf{B}'_T(\partial\Omega),\mathbf{B}_T(\partial\Omega)} = \int_\Omega \boldsymbol{u}\cdot\mathbf{curl}(\boldsymbol{v})\,dx - \int_\Omega \boldsymbol{v}\cdot\mathbf{curl}(\boldsymbol{u})\,dx.$$

Proof. We have $\mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}) \in \mathbf{B}'_{\mathbf{curl}}(\Omega)$ hence $\mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}) \in \mathbf{B}'_T(\Omega)$ because $\mathbf{B}_T(\Omega) \subset \mathbf{B}_{\mathbf{curl}}(\Omega)$. And by (4) we know we have :

$$\langle \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}), \mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{v}) \rangle_{\mathbf{B}'_{\mathbf{curl}}(\partial\Omega),\mathbf{B}_{\mathbf{curl}}(\partial\Omega)} = \int_\Omega \boldsymbol{u}\cdot\mathbf{curl}(\boldsymbol{v})\,dx - \int_\Omega \boldsymbol{v}\cdot\mathbf{curl}(\boldsymbol{u})\,dx.$$

And we have :

$$\begin{aligned}\langle \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}), \mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{v}) \rangle_{\mathbf{B}'_{\mathbf{curl}}(\partial\Omega),\mathbf{B}_{\mathbf{curl}}(\partial\Omega)} = \langle \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}), (\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{v}))_T \rangle_{\mathbf{B}'_{\mathbf{curl}}(\partial\Omega),\mathbf{B}_{\mathbf{curl}}(\partial\Omega)} +\\ \langle \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}), (\,\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{v}))_{\perp} \rangle_{\mathbf{B}'_{\mathbf{curl}}(\partial\Omega),\mathbf{B}_{\mathbf{curl}}(\partial\Omega)}\end{aligned}$$

And by definition of $\mathbf{B}_{\perp}(\partial\Omega)$, the second term is zero. It remains :

$$\begin{aligned}\langle \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}), \mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{v}) \rangle_{\mathbf{B}'_{\mathbf{curl}}(\partial\Omega),\mathbf{B}_{\mathbf{curl}}(\partial\Omega)} &= \langle \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}), (\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{v}))_T \rangle_{\mathbf{B}'_{\mathbf{curl}}(\partial\Omega),\mathbf{B}_{\mathbf{curl}}(\partial\Omega)}\\ &= \langle \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{u}), (\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{v}))_T \rangle_{\mathbf{B}'_T(\partial\Omega),\mathbf{B}_T(\partial\Omega)}.\end{aligned}$$

□

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^N$ verifying $(\mathcal{P}1)$. Then the tangential trace operator $\mathbf{Tr}_{\mathrm{T}} : \mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{B}'_T(\partial\Omega)$ is surjective. Moreover, the space $\mathbf{B}'_T(\partial\Omega)$ endowed with the norm $\|\boldsymbol{e}\|_{\mathbf{B}'_T(\partial\Omega)} := \inf\{\|\boldsymbol{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)},\ \boldsymbol{u} \in \mathbf{H}(\mathbf{curl},\Omega),\ s.t.\ (\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{u}))_T = \boldsymbol{e}\ q.e.\}$ is a Hilbert space.*

Proof. To proove the fact that $\mathbf{B}'_T(\partial\Omega)$ endowed with the norm $\|.\|_{\mathbf{B}'_T(\partial\Omega)}$ is a Hilbert space, it is similar to the proof of Theorem 5.2

To prove surjectivity, for $\mu \in \boldsymbol{B}'_T(\partial\Omega)$, we need to find $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{curl},\Omega)$ such that

$$\mathrm{Tr}_T(\boldsymbol{u}) = \mu.$$

To this aim, we consider the following problem:

$$\begin{cases} \mathrm{curl}(\mathrm{curl}\ \boldsymbol{w}) + \boldsymbol{w} = 0 & \text{in } \Omega, \\ \mathrm{Tr}_T(\mathrm{curl}\ \boldsymbol{w}) = \mu & \text{on } \partial\Omega. \end{cases}$$

Using the Stokes formula, the weak solution of this problem is understood in the sense of the variational formulation on $\boldsymbol{H}(\mathrm{curl}, \Omega)$, given by:

$$\forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, \Omega), \quad (\boldsymbol{w}, \boldsymbol{v})_{\boldsymbol{H}(\mathrm{curl};\Omega)} = \langle \mathrm{Tr}_T(\mathrm{curl}\ \boldsymbol{w}), \mathrm{Tr}_T(\boldsymbol{v}) \rangle_{\boldsymbol{B}'_T(\partial\Omega), \boldsymbol{B}_T(\partial\Omega)}. \tag{9}$$

Indeed,

$$\forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, \Omega), \quad \int_\Omega (\boldsymbol{u} \cdot \mathrm{curl}\, \boldsymbol{v} - \boldsymbol{v} \cdot \mathrm{curl}\, \boldsymbol{u})\, d\boldsymbol{x} = \langle \mathrm{Tr}_T(\boldsymbol{u}), \mathrm{Tr}_T(\boldsymbol{v}) \rangle_{\boldsymbol{B}'_T(\partial\Omega), \boldsymbol{B}_T(\partial\Omega)}.$$

Hence,

$$(\mathrm{curl}\ \boldsymbol{w}, \mathrm{curl}\ \boldsymbol{v}) - (\mathrm{curl}\,\mathrm{curl}\ \boldsymbol{w}, \boldsymbol{v}) = \langle \mathrm{Tr}_T(\mathrm{curl}\ \boldsymbol{w}), \mathrm{Tr}_T(\boldsymbol{v}) \rangle_{\boldsymbol{B}'_T(\partial\Omega)}.$$

Or,

$$\mathrm{curl}(\mathrm{curl}\ \boldsymbol{w}) + \boldsymbol{w} = 0.$$

Finally,

$$(\mathrm{curl}\ \boldsymbol{w}, \mathrm{curl}\ \boldsymbol{v}) + (\boldsymbol{w}, \boldsymbol{v}) = \langle \mathrm{Tr}_T(\mathrm{curl}\ \boldsymbol{w}), \mathrm{Tr}_T(\boldsymbol{v}) \rangle_{\boldsymbol{B}'_T(\partial\Omega)}.$$

According to the Lax-Milgram theorem, (9) admits a unique solution $\boldsymbol{w}$ of:

$$\forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, \Omega), \quad (\boldsymbol{w}, \boldsymbol{v})_{\boldsymbol{H}(\mathrm{curl};\Omega)} = \langle \mu, \mathrm{Tr}_T(\boldsymbol{v}) \rangle_{\boldsymbol{B}'_T(\partial\Omega)}.$$

By taking $\boldsymbol{u} = \mathrm{curl}\ \boldsymbol{w}$ in $\boldsymbol{H}(\mathrm{curl}, \Omega)$, we have the surjectivity of Tr_T from $\boldsymbol{H}(\mathrm{curl}, \Omega)$ onto $\boldsymbol{B}'_T(\partial\Omega)$.

□

5 Tangential component of the trace operator

Theorem 5.1 (Tangential component of the trace operator)**.**

Let $\Omega \subset \mathbb{R}^N$ verifying $(\mathcal{P}1)$. There exists a bounded operator called the tangential component of the trace $\boldsymbol{\pi}_T : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{B}'_\perp(\partial\Omega)$ which is defined as the following for $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$:

$$\langle \boldsymbol{\pi}_T(\boldsymbol{u}), (\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{v}))_\perp \rangle_{\mathbf{B}'_\perp(\partial\Omega), \mathbf{B}_\perp(\partial\Omega)} = \int_\Omega \mathbf{curl}(\boldsymbol{u}) \cdot \boldsymbol{v}\, dx - \int_\Omega \mathbf{curl}(\boldsymbol{v}) \cdot \boldsymbol{u}\, dx. \tag{10}$$

The kernel of $\boldsymbol{\pi}_T$ is $\mathbf{H}_0(\mathbf{curl}, \Omega)$.

Proof. We show this is well defined the same way as we did for the previous definition of the tangential trace operator in theorem 4.1, relying on lemma 4.1, and do the same thing as in the proof of theorem 4.1 to prove its boundedness. The fact that the kernel of $\boldsymbol{\pi_T}$ is $\mathbf{H}_0(\mathbf{curl}, \Omega)$ just comes from lemma 4.1 as well. □

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^N$ verifying $(\mathcal{P}1)$. Then the tangential component of the trace operator $\boldsymbol{\pi}_T : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{B}'_\perp(\partial\Omega)$ is surjective. Moreover, the space $\mathbf{B}'_\perp(\partial\Omega)$ endowed with the norm $\|\boldsymbol{e}\|_{\mathbf{B}'_\perp(\partial\Omega)} := \inf\{\|\boldsymbol{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)},\ \boldsymbol{u} \in \mathbf{H}(\mathbf{curl}, \Omega),\ s.t.\ (\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{u}))_\perp = \boldsymbol{e}\ q.e.\}$ is a Hilbert space.*

Proof. To proove the surjectivity it is similar to the proof of Theorem 4.2 We consider the following problem:

$$\begin{cases} \mathrm{curl}(\mathrm{curl}\ \boldsymbol{w}) + \boldsymbol{w} = 0 & \text{in } \Omega, \\ (\mathbf{Tr}_{\mathbf{curl}}(\boldsymbol{w}))_\perp = \boldsymbol{e} & \text{on } \mathbf{B}_\perp(\partial\Omega). \end{cases}$$

Let us define the space

$$\mathbf{H}_{(\mathrm{curl},\Omega);e} = \{\boldsymbol{u} \in \mathbf{H}(\mathrm{curl},\Omega) \mid (\mathbf{Tr_{curl}}(\boldsymbol{u}))_\perp = \boldsymbol{e} \text{ q.e}\}.$$

The initial problem can be reformulated using the following variational formulation:

$$\forall \boldsymbol{v} \in \mathbf{H}_{(\mathrm{curl},\Omega;e)}, \quad (\boldsymbol{w},\boldsymbol{v})_{\mathbf{H}(\mathrm{curl};\Omega)} = -\langle \boldsymbol{\pi}_T(\mathrm{curl}(\boldsymbol{w})), \mathrm{Tr}_\perp(\boldsymbol{v})\rangle_{\mathbf{B}'_\perp(\partial\Omega)}.$$

Since $\boldsymbol{v} \in \mathbf{H}_{(\mathrm{curl},\Omega;e)}$, we know that $\mathrm{Tr}_\perp(\boldsymbol{v}) = (\mathbf{Tr_{curl}}(\boldsymbol{w}))_\perp$. Therefore, using identity (12), the variational formulation becomes:

$$\forall \boldsymbol{v} \in \mathbf{H}(\mathrm{curl},\Omega;e), \quad (\boldsymbol{w},\boldsymbol{v})_{\mathbf{H}(\mathrm{curl};\Omega)} = \int_\Omega \mathrm{curl}(\boldsymbol{w})\cdot\mathrm{curl}(\boldsymbol{w})\,dx - \int_\Omega \mathrm{curl}(\mathrm{curl}(\boldsymbol{w}))\cdot\boldsymbol{w}\,dx.$$

Adding the fact that $\mathrm{curl}(\mathrm{curl}\ \boldsymbol{w}) + \boldsymbol{w} = 0$, we obtain:

$$\forall \boldsymbol{v} \in \mathbf{H}(\mathrm{curl},\Omega;e), \quad (\boldsymbol{w},\boldsymbol{v}-\boldsymbol{w})_{\mathbf{H}(\mathrm{curl};\Omega)} = 0.$$

In order to apply Stampacchia's theorem, we need to prove that $\mathbf{H}_{(\mathrm{curl},\Omega;e)}$ is convex and closed, which follows from the linearity and continuity of the trace operator $\mathbf{Tr_{curl}}$.

Hence, by Stampacchia's theorem, there exists a unique solution

$$\exists! \mathbf{w} \in \mathbf{H}_{(\mathrm{curl},\Omega;e)}(\Omega), \quad \forall \mathbf{v} \in \mathbf{H}_{(\mathrm{curl},\Omega;e)}, \quad (\mathbf{w},\mathbf{v}-\mathbf{w})_{\mathbf{H}(\mathrm{curl},\Omega)} = 0.$$

Moreover, $\mathbf{w}$ is the unique minimizer over $\mathbf{H}_{(\mathrm{curl},\Omega;e)}$ of the functional $\|\mathbf{v}\|^2_{\mathbf{H}(\mathrm{curl},\Omega)}$.

If we denote $\|\mathbf{w}\|_{\mathbf{H}(\mathrm{curl},\Omega)} = \|\boldsymbol{e}\|_{\mathbf{B}_\perp(\partial\Omega)}$, then, since $\mathbf{H}(\mathrm{curl},\Omega)$, endowed with the norm $\|\cdot\|_{\mathbf{H}(\mathrm{curl},\Omega)}$, is a Hilbert space, it follows that $\mathbf{B}_\perp(\partial\Omega)$, endowed with the norm $\|\cdot\|_{\mathbf{B}_\perp(\partial\Omega)}$, is also a Hilbert space. Consequently, so is $\mathbf{B}'_\perp(\partial\Omega)$.

□

6 Tangential differential operators

We shall start with a remark, that for every $u \in H^1(\Omega)$, it holds that $\mathbf{curl}(\mathbf{u}) \in \mathbf{H}(\mathbf{curl},\Omega)$, because $\mathbf{curl}(\boldsymbol{\nabla} u) = 0$. This allows us to define the tangential gradient operator. It is hence very important here that we are working on $\mathbf{B}_1(\partial\Omega)$, and not $\mathbf{B_{curl}}(\partial\Omega)$

Definition 6.1. *Let $\Omega \subset \mathbb{R}^N$ be a domain verifying ($\mathcal{P}1$). We define the tangential gradient operator $\boldsymbol{\nabla}_T$;*

$$\left| \begin{array}{ccc} \mathrm{B}_1(\partial\Omega) & \longrightarrow & \mathbf{B}'_\perp(\partial\Omega) \\ \mathrm{Tr}(u) & \longmapsto & \boldsymbol{\nabla}_T(\mathrm{Tr}(u)) := \boldsymbol{\pi}_T(\boldsymbol{\nabla} u). \end{array} \right.$$

Proof. First, we will justify that this operator is well defined. That is, we have to prove that $\mathrm{Tr}(u) = 0$ implies $\boldsymbol{\pi}_T(\boldsymbol{\nabla} u) = 0$; let $u \in H^1(\Omega)$ be such that $\mathrm{Tr}(u) = 0$. We already know that this is equivalent to $u \in H^1_0(\Omega)$, so we can find a sequence $(u_n) \subset \mathcal{D}(\Omega)$ that converges to u in $H^1(\Omega)$. It is then easy to see that $(\boldsymbol{\nabla} u_n) \subset \mathcal{D}(\Omega)^N$ converges to $\boldsymbol{\nabla} u$ in $\mathbf{H}(\mathbf{curl},\Omega)$, because all the curls are 0. Hence we have proved that $\mathrm{Tr}(u) = 0 \Rightarrow \boldsymbol{\pi}_T(\boldsymbol{\nabla} u) = 0$ so the operator is well defined. □

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^N$ be a domain verifying ($\mathcal{P}1$). Then the tangential gradient operator is bounded and its image verifies $\boldsymbol{\nabla}_T(\mathrm{B}_1(\partial\Omega)) \subset \mathbf{B}'_\perp(\partial\Omega)$.*

Proof. As we have done in the proof of theorem 4.1, we notice that one way to write our tangential gradient operator is using the right-inverse of the trace on $H^1(\Omega)$. Indeed, if $E : \mathrm{B}_1(\partial\Omega) \to H^1(\Omega)$ is this operator, it can be easily proved that one way to define $\boldsymbol{\nabla}_T$ is with the following, for $f \in \mathrm{B}_1(\partial\Omega)$: $\boldsymbol{\nabla}_T(f) = \boldsymbol{\pi}_T(\boldsymbol{\nabla} E(f))$. Then it is clear that $\boldsymbol{\nabla}_T$ is bounded seeing that E is bounded on $\mathrm{B}_1(\partial\Omega)$, $u \mapsto \boldsymbol{\nabla} u$ is bounded from $H^1(\Omega)$ to $\mathbf{H}(\mathbf{curl},\Omega)$ and according to theorem 5.1, $\boldsymbol{\pi}_T$ is bounded from $\mathbf{H}(\mathbf{curl},\Omega)$ to $\mathbf{B}'_\perp(\partial\Omega)$. □

Now we have a boundary tangential gradient operator, we can also define a tangential divergence operator by analogy with the classic results.

Definition 6.2. *Let $\Omega \subset \mathbb{R}^N$ be a domain verifying $(\mathcal{P}1)$. Then we define the tangential divergence operator* $\operatorname{div}_{\partial\Omega} : \mathbf{B}_{\perp}(\partial\Omega) \to \mathrm{B}_1'(\partial\Omega)$ *as the adjoint operator of* $-\boldsymbol{\nabla}_T$*.*

We now have a bounded tangential divergence operator that verifies the classical relation between gradient and divergence. That is, for all $\phi \in H^1(\Omega)$, for all $\boldsymbol{\psi} \in \mathbf{B}_1(\partial\Omega)$:

$$\langle -\boldsymbol{\nabla}_T(\mathrm{Tr}(\phi)), \boldsymbol{\psi}\rangle_{\mathbf{B}_{\perp}'(\partial\Omega),\mathbf{B}_{\perp}(\partial\Omega)} = \langle \operatorname{div}_{\partial\Omega}(\boldsymbol{\psi}), \mathrm{Tr}(\phi)\rangle_{\mathbf{B}_1'(\partial\Omega),\mathbf{B}_1(\partial\Omega)}.$$

Definition 6.3. *Let $\Omega \subset \mathbb{R}^N$ be a domain verifying $(\mathcal{P}1)$. We define the tangential curl operator* $\mathbf{curl}_{\partial\Omega}$*:*

$$\left|\begin{array}{ccc} \mathrm{B}_1(\partial\Omega) & \longrightarrow & \mathbf{B}_T'(\partial\Omega) \\ \mathrm{Tr}(u) & \longmapsto & \mathbf{curl}_{\partial\Omega}(\mathrm{Tr}(u)) := \mathbf{Tr}_{\mathrm{T}}(\boldsymbol{\nabla} u). \end{array}\right.$$

Proof. To show this is well defined, we proceed exactly as we did to deal with the tangential gradient operator in definition 6.1. □

Lemma 6.2. *Let $\Omega \subset \mathbb{R}^N$ be a domain verifying $(\mathcal{P}1)$. The the tangential curl operator is bounded.*

Proof. We prove this the same exact way we did to prove lemma 6.1. □

7 Decomposition of vector fields

Here we extend some already existing results about the decomposition of L^2 vector fields (see [10]), and that are very commonly used by physicists.

Lemma 7.1. *Let $\Omega \subset \mathbb{R}^N$ be a Sobolev admissible domain. Then there exists some $C > 0$ such that for every $u \in \mathrm{H}^1(\Omega)$ such that $\int_\Omega u\, dx = 0$, we have $\|u\|_{\mathrm{L}^2(\Omega)} \leq C\|\,\mathbf{grad}\, u\|_{\mathbf{L}^2(\Omega)}$.*

Proof. Suppose for contradiction that there exists some sequence $(u_k) \subset \mathrm{H}^1(\Omega)$ that verifies for every k:

- $\int_\Omega u_k\, dx = 0$.
- $\int_\Omega |u_k|^2\, dx = 1$.
- $\int_\Omega \|\,\mathbf{grad}\, u_k\|^2\, dx \leq \frac{1}{k}$.

Hence (u_k) is bounded in $\mathrm{H}^1(\Omega)$ so we can extract a subsequence (u_{k_j}) of (u_k) that converges weakly to some $u \in \mathrm{H}^1(\Omega)$. And by proposition 2 of [11], we know that (u_{k_j}) converges strongly to u in $\mathrm{L}^2(\Omega)$. Because (u_{k_j}) converges strongly in $\mathrm{L}^2(\Omega)$, we can deduce that $\|u\|_{\mathrm{L}^2(\Omega)} = 1$. Besides, because (u_{k_j}) converges strongly in $\mathrm{L}^2(\Omega)$, we know it also converges weakly; hence, $\langle u_{k_j}, 1\rangle_{\mathrm{L}^2(\Omega)} \to \langle u, 1\rangle_{\mathrm{L}^2(\Omega)}$, which implies $\langle u, 1\rangle_{\mathrm{L}^2(\Omega)} = \int_\Omega u\, dx = 0$. Finally we know that:

$$\begin{aligned} \|u\|_{\mathrm{H}^1(\Omega)} &\leq \liminf \|u_{k_j}\|_{\mathrm{H}^1(\Omega)} \\ \|u\|_{\mathrm{H}^1(\Omega)} &\leq 1. \end{aligned}$$

Because $\int_\Omega \|\,\mathbf{grad}\, u_{k_j}\|^2\, dx \leq \frac{1}{k_j} \xrightarrow[j\to+\infty]{} 0$. And on the other hand $\|u\|_{\mathrm{H}^1(\Omega)} = \sqrt{1 + \int_\Omega \|\,\mathbf{grad}\, u\|^2\, dx}$; this implies that $\int_\Omega \|\,\mathbf{grad}\, u\|^2\, dx = 0$. So u has to be constant; but then it cannot verify both $\int_\Omega u\, dx = 0$ and $\int_\Omega |u|^2\, dx = 1$.

□

Definition 7.1. *Let $\Omega \subset \mathbb{R}^N$ be a domain. We define the space $\mathbf{H}(\Omega) = \{\boldsymbol{u} \in \mathbf{H}_0(\operatorname{div}, \Omega),\ s.t\ \operatorname{div}(\boldsymbol{u}) = 0\}$. $\mathbf{H}(\Omega)$ is a closed subspace of $\mathbf{L}^2(\Omega)$, hence it holds $\mathbf{L}^2(\Omega) = \mathbf{H}(\Omega) \oplus \mathbf{H}^{\perp}(\Omega)$. Here $\mathbf{H}^{\perp}(\Omega)$ is the orthogonal of $\mathbf{H}(\Omega)$ in the L^2 sense.*

Proof. Let $(\boldsymbol{f}_n) \subset \mathbf{H}(\Omega)$ be a sequence that converges in $\mathbf{L}^2(\Omega)$ to some $\boldsymbol{f}$. Let us prove that $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$. Because the divergence operator $\operatorname{div} : \mathbf{L}^2(\Omega) \to \mathbf{H}^{-1}(\Omega)$ is continuous, we directly obtain that $\operatorname{div}(\boldsymbol{f}) = 0$. Now because $(\boldsymbol{f}_n)$ converges to $\boldsymbol{f}$ in $\mathbf{L}^2(\Omega)$, it is bounded by some constant $K > 0$ in $\mathbf{L}^2(\Omega)$. But because for every n it holds that $\operatorname{div}(\boldsymbol{f}_n) = 0$, this is equivalent to saying that $(\boldsymbol{f}_n)$ is bounded in $\mathbf{H}(\operatorname{div}, \Omega)$. Now because $\mathbf{H}(\operatorname{div}, \Omega)$ is a Hilbert space, this implies that there exists some subsequence $(\boldsymbol{g}_n)$ of $(\boldsymbol{f}_n)$ that converges weakly in $\mathbf{H}(\operatorname{div}, \Omega)$. By uniqueness of the weak limit, this limit has to be $\boldsymbol{f}$. Besides, $\mathbf{H}_0(\operatorname{div}, \Omega)$ is a convex strongly closed subset of $\mathbf{H}(\operatorname{div}, \Omega)$, hence it is also weakly closed. This implies that $\boldsymbol{f} \in \mathbf{H}_0(\operatorname{div}, \Omega)$. In conclusion we have $\boldsymbol{f} \in \mathbf{H}(\Omega)$. So this space is closed in $\mathbf{L}^2(\Omega)$, and hence the decomposition of $\mathbf{L}^2(\Omega)$. □

Lemma 7.2. *Let $\Omega \subset \mathbb{R}^N$. Then $\mathbf{H}(\Omega)$ is a Hilbert space for the norm of $\mathbf{H}(\operatorname{div}, \Omega)$.*

Proof. By definition $\mathbf{H}_0(\operatorname{div}, \Omega)$ is closed in $\mathbf{H}(\operatorname{div}, \Omega)$, and because the divergence operator is continuous on $\mathbf{H}(\operatorname{div}, \Omega)$, its kernel is closed in $\mathbf{H}(\operatorname{div}, \Omega)$ as well. Hence $\mathbf{H}(\Omega)$ is closed in $\mathbf{H}(\operatorname{div}, \Omega)$ as the intersection of two closed subspaces, and it is thus a Hilbert space for the norm of $\mathbf{H}(\operatorname{div}, \Omega)$. □

We can also notice that because for every $\boldsymbol{u} \in \mathbf{H}(\Omega)$, we have $\operatorname{div}(\boldsymbol{u}) = 0$, the $\mathbf{H}(\operatorname{div})$ norm on $\mathbf{H}(\Omega)$ is exactly the $\mathbf{L}^2(\Omega)$.

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^N$ be a domain verifying $(\mathcal{P}2)$. Then we have $\mathbf{H}^{\perp}(\Omega) = \{\mathbf{grad}\, q,\ q \in \mathbf{H}^1(\Omega)\}$.*

Proof. Let us begin by justifying that $X = \{\mathbf{grad}\, q,\ q \in \mathbf{H}^1(\Omega)\}$ is a closed subspace of $\mathbf{L}^2(\Omega)$. Let $(p_k) \subset \mathrm{H}^1(\Omega)$ be such that $\mathbf{grad}\, p_k \xrightarrow[k\to\infty]{L^2} \boldsymbol{v} \in \mathbf{L}^2(\Omega)$. Because the gradient of a constant is zero, we can assume that for every k, we have $\int_\Omega p_k\, dx = 0$. Hence the Poincaré-Wirtinger inequality (lemma 7.1, which is verified because Ω is Sobolev admissible) assures us that there exists some constant C such that : $\|p_k\|_{\mathrm{L}^2(\Omega)} \le C \|\mathbf{grad}\, p_k\|_{\mathbf{L}^2(\Omega)}$. Because $(\mathbf{grad}\, p_k)$ converges in $\mathbf{L}^2(\Omega)$, this assures us that (p_k) is bounded in $\mathrm{H}^1(\Omega)$. By the Kakutani theorem, we can then extract some subsequence (q_k) that converges weakly in $\mathrm{H}^1(\Omega)$ to some q. But we also know that $\mathbf{grad}\, p_k \xrightarrow[k\to\infty]{\mathbf{L}^2} \boldsymbol{v}$, so we also have $\mathbf{grad}\, q_k \xrightarrow[k\to\infty]{\mathbf{L}^2} \boldsymbol{v}$. By the uniqueness of the weak limit, we deduce that $\mathbf{grad}\, q = \boldsymbol{v}$, so X is indeed a closed subspace of $\mathbf{L}^2(\Omega)$.
Now that this is clear, to prove that $\mathbf{H}^{\perp}(\Omega) = X$, we can rather prove that $X^{\perp} = \mathbf{H}(\Omega)$. First, let $\boldsymbol{u} \in \mathbf{H}(\Omega)$. By formula (1) and definition of $\mathbf{H}(\Omega)$, we obtain:

$$\int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\, q\, dx = 0.$$

Therefore $\mathbf{H}(\Omega) \subset X^{\perp}$. Conversely, let $\mathbf{u} \in L^2(\Omega)^N$ with

$$(\mathbf{u}, \mathbf{grad}\, q) = 0, \quad \forall q \in H^1(\Omega)$$

By choosing q in $\mathcal{D}(\Omega)$, this implies that $\operatorname{div} \mathbf{u} = 0$. Hence $\mathbf{u} \in \mathbf{H}(\operatorname{div}, \Omega)$. Therefore we can apply formula (1) which gives $\mathrm{Tr}_n(\boldsymbol{u}) = 0$. In view of Theorem 2.2, this means that $\mathbf{u} \in H$. Therefore $X^{\perp} \subset H$. □

Definition 7.2. *Let $\Omega \subset \mathbb{R}^N$ be a domain. We define $\mathrm{L}^2_0(\Omega) := \{f \in \mathrm{L}^2(\Omega),\ s.t.\ \int_\Omega f\, dx = 0\}$.*

Lemma 7.3. *Let $\Omega \subset \mathbb{R}^N$ be a domain. Then $\mathcal{D}(\Omega) \cap \mathrm{L}^2_0(\Omega)$ is dense in $\mathrm{L}^2_0(\Omega)$.*

Proof. Let $f \in \mathrm{L}^2_0(\Omega)$ and $(f_n) \subset \mathcal{D}(\Omega)$ a sequence that converges to f in $\mathrm{L}^2(\Omega)$. Then let $\chi \in \mathcal{D}(\Omega)$ be a function that verifies $\int_\Omega \chi\, dx = 1$. We define the sequence $g_n = f_n - (\frac{1}{|\Omega|} \int_\Omega f_n\, dx)\chi$. It is clear that for every n, we have $g_n \in \mathcal{D}(\Omega) \cap \mathrm{L}^2_0(\Omega)$ and we directly obtain that (g_n) converges to f in $\mathrm{L}^2_0(\Omega)$ hence the result. □

Theorem 7.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Sobolev admissible domain. Then the range of the divergence operator* $\operatorname{div} : \mathbf{H}_0^1(\Omega) \to \mathrm{L}^2(\Omega)$ *is exactly* $\mathrm{L}_0^2(\Omega) := \{f \in \mathrm{L}^2(\Omega),\ s.t.\ \int_\Omega f\,dx = 0\}$.

Proof. Let $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$. Then we can deduce by using formula (1) that $\int_\Omega \operatorname{div}(\boldsymbol{u})\,dx = 0$, hence $\operatorname{div}(\boldsymbol{u}) \in \mathrm{L}_0^2(\Omega)$.

For the reciprocal inclusion, let $f \in \mathrm{L}_0^2(\Omega)$. Let (Ω_n) be an increasing sequence of Lipschitz domains which converges to Ω. Then we know by Corollary 2.1 of [10] that the range of the gradient operator from $\mathrm{L}^2(\Omega_n)$ to $\mathrm{H}^{-1}(\Omega_n)$ is closed (because these domains are Lipschitz and bounded). Knowing this, Theorem 3.1 of [1] assures us that the divergence operator $\operatorname{div} : \mathrm{H}_0^1(\Omega_n) \to \mathrm{L}_0^2(\Omega_n)$ is onto. Let us define :

$$g_n = f\mathbb{1}_{\Omega_n} + \frac{1}{|\Omega_n|}\int_{\Omega\setminus\Omega_n} f\,dx.$$

It can be easily verified that $g_n\big|_{\Omega_n} \in \mathrm{L}_0^2(\Omega_n)$. Hence there exists some $\boldsymbol{u_n}$ in $\mathbf{H}_0^1(\Omega_n)$ such that $\operatorname{div}(\boldsymbol{u}_n) = g_n$ on Ω_n and $\|\boldsymbol{u}_n\|_{\mathbf{H}_0^1(\Omega_n)} \leq C_n\|g_n\|_{\mathrm{L}^2(\Omega_n)}$. We have justified in lemma 2.1 that $\boldsymbol{u_n}$ extended by zero to Ω is in $\mathbf{H}_0^1(\Omega)$. With a slight abuse of notation, we shall denote u_n for both the elements of $\mathbf{H}_0^1(\Omega_n)$ or $\mathbf{H}_0^1(\Omega)$. We thus have for every n the following :

$$\begin{cases} \operatorname{div}(\boldsymbol{u}_n) = g_n\mathbb{1}_{\Omega_n} \text{ on } \Omega \\ \|\boldsymbol{u}_n\|_{\mathbf{H}_0^1(\Omega)} \leq C_n\|g_n\|_{\mathrm{L}^2(\Omega_n)} \leq C_n\|g_n\|_{\mathrm{L}^2(\Omega)}. \end{cases} \tag{11}$$

We then compute $\|g_n\|_{\mathrm{L}^2(\Omega)}$, with the intent of proving that this sequence is bounded:

$$\|g_n\|^2_{\mathrm{L}^2(\Omega)} = \|f\mathbb{1}_{\Omega_n}\|^2_{\mathrm{L}^2(\Omega)} + \frac{|\Omega|}{|\Omega_n|^2}\Big(\int_{\Omega\setminus\Omega_n} f\,dx\Big)^2 + \frac{2}{|\Omega_n|}\Big(\int_{\Omega\setminus\Omega_n} f\,dx\Big)\Big(\int_{\Omega_n} f\,dx\Big).$$

Starting with the first term; we know that $\|f\mathbb{1}_{\Omega_n}\|^2_{\mathrm{L}^2(\Omega)} \leq \|f\|^2_{\mathrm{L}^2(\Omega)}$, so this sequence is bounded. Then we deal with the second term. First notice that because the sequence (Ω_n) is increasingly convergent to Ω, we know that for every n, we have $\frac{1}{|\Omega_n|} \leq \frac{1}{|\Omega_0|}$. Hence we will not account for the $\frac{1}{|\Omega_n|}$ terms as those are bounded.

$$\begin{aligned} \Big(\int_{\Omega\setminus\Omega_n} f\,dx\Big)^2 &\leq \Big(\int_{\Omega\setminus\Omega_n} |f|\,dx\Big)^2 \\ &\leq \Big(\int_{\Omega} |f|\,dx\Big)^2 \\ &\leq \|f\|^2_{\mathrm{L}^1(\Omega)}. \end{aligned}$$

And because Ω is bounded we know that $\mathrm{L}^2(\Omega) \subset \mathrm{L}^1(\Omega)$ both algebraically and topologically. And we already justified that $\|f\|_{\mathrm{L}^2(\Omega)} < +\infty$ hence $\|f\|_{\mathrm{L}^1(\Omega)} < +\infty$ as well. Now we deal with the third term. We just proved that the quantity $(\int_{\Omega\setminus\Omega_n} f\,dx)$ is bounded, so it only remains to prove that $(\int_{\Omega_n} f\,dx)$ is bounded as well:

$$\begin{aligned} \Big|\int_{\Omega_n} f\,dx\Big| &\leq \int_{\Omega_n} |f|\,dx \\ &= \int_{\Omega} |f|\mathbb{1}_{\Omega_n}\,dx \\ &\underbrace{\leq}_{\text{Cauchy-Schwarz}} \|\mathbb{1}_{\Omega_n}\|_{\mathrm{L}^2(\Omega)}\|f\|_{\mathrm{L}^2(\Omega)}. \end{aligned}$$

And this last thing is bounded because $\|f\|_{\mathrm{L}^2(\Omega)} < +\infty$ and $\|\mathbb{1}_{\Omega_n}\|_{\mathrm{L}^2(\Omega)} \xrightarrow[n\to\infty]{} |\Omega|$. Finally we know that $(\|g_n\|_{L^2(\Omega)})$ is bounded. Besides, in (11), recall that the constant C_n can be computed by:

$$\frac{1}{C_n} = \sup_{\boldsymbol{v}\in\mathbf{H}^1_0(\Omega_n),\ \|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega_n)}=1} \|\operatorname{div}(\boldsymbol{v})\|_{L^2(\Omega_n)}.$$

Now, relying on lemma 2.1, it can be observed that the only difference between the inf defining C_n and the one defining C_{n+1} is the set on which it is taken. Because $\Omega_n \subset \Omega_{n+1}$, all the functions of $\mathbf{H}^1_0(\Omega_n)$ are also in $\mathbf{H}^1_0(\Omega_{n+1})$ (when extended by zero). Following up on this observation we obtain that $C_n \geq C_{n+1}$. Hence we have for every n: $C_n \leq C_0$. Now recall (11): we know that both (C_n) and $(\|g_n\|_{L^2(\Omega)})$ are bounded hence $(\boldsymbol{u}_n)$ is bounded in $\mathbf{H}^1_0(\Omega)$. Now by Kakutani's theorem we know that we can extract a subsequence $(\boldsymbol{u}_{k_n})$ of $(\boldsymbol{u}_n)$ that converges weakly to some $\boldsymbol{u}$ in $\mathbf{H}^1_0(\Omega)$. Because $\operatorname{div} : \mathbf{H}^1_0(\Omega) \to L^2(\Omega)$ is strongly continuous, we know it is also weakly continuous so $\operatorname{div}(\boldsymbol{u}_{k_n}) \xrightharpoonup[n\to\infty]{} \operatorname{div}(\boldsymbol{u})$. And besides, for every n, we have:

$$\begin{aligned}\operatorname{div}(\boldsymbol{u}_{k_n}) &= g_{k_n}\mathbb{1}_{\Omega_{k_n}} \\ &= f\mathbb{1}_{\Omega_{k_n}} + \frac{\mathbb{1}_{\Omega_{k_n}}}{|\Omega_{k_n}|}\int_{\Omega\setminus\Omega_{k_n}} f\,dx.\end{aligned}$$

First, observe that by Cauchy Schwarz, it holds:

$$\begin{aligned}|\frac{\mathbb{1}_{\Omega_{k_n}}}{|\Omega_{k_n}|}\int_{\Omega\setminus\Omega_{k_n}} f\,dx| &\leq \frac{1}{|\Omega_0|}|\int_{\Omega\setminus\Omega_{k_n}} f\,dx| \\ &\leq \frac{1}{|\Omega_0|}\int_{\Omega\setminus\Omega_{k_n}} |f|\,dx \\ &\leq \frac{1}{|\Omega_0|}\|f\|_{L^2(\Omega)} \underbrace{\|\mathbb{1}_{\Omega\setminus\Omega_{k_n}}\|_{L^2(\Omega)}.}_{\xrightarrow[n\to\infty]{}0}\end{aligned}$$

So the sequence $(\frac{\mathbb{1}_{\Omega_{k_n}}}{|\Omega_{k_n}|}\int_{\Omega\setminus\Omega_{k_n}} f\,dx)$ converges simply to 0 on Ω. Because we have the domination $|\frac{\mathbb{1}_{\Omega_{k_n}}}{|\Omega_{k_n}|}\int_{\Omega\setminus\Omega_{k_n}} f\,dx| \leq \frac{1}{|\Omega_0|}|\int_\Omega f\,dx|$, we can use the dominated convergence theorem to deduce that it also converges both strongly and weakly in $L^2(\Omega)$ to 0. Now we shall prove that the sequence $(f\mathbb{1}_{\Omega_{k_n}})$ converges weakly to f in $L^2(\Omega)$ to f and the result will be established. So let $h \in L^2(\Omega)$:

$$\begin{aligned}|\langle f\mathbb{1}_{\Omega_{k_n}}, h\rangle - \langle f, h\rangle| &= |\int_\Omega (f\mathbb{1}_{\Omega_{k_n}}h - fh)\,dx| \\ &\leq \int_\Omega |fh(1-\mathbb{1}_{\Omega_{k_n}})|\,dx \\ &\leq \|fh\|_{L^2(\Omega)} \underbrace{\|1-\mathbb{1}_{\Omega_{k_n}}\|_{L^2(\Omega)}}_{\xrightarrow[n\to+\infty]{}0}\end{aligned}$$

Because (Ω_n) converges to Ω in the sense of characteristic functions. So $(f\mathbb{1}_{\Omega_{k_n}})$ converges weakly to f and we have the result by uniqueness of the weak limit. □

Lemma 7.4. *Let Ω be a bounded Sobolev admissible domain and ω a sub domain of Ω of strictly positive Lebesgue measure. There exists a constant $C > 0$ such that for every $p \in L^2(\Omega)$, we have $\|p\|_{L^2(\Omega)} \leq C(\|p\|_{L^2(\omega)} + \|\mathbf{grad}\, p\|_{H^{-1}(\Omega)})$.*

Proof. We take Ω_n an increasing sequence of Sobolev admissible Lipschitz sub-domains that converges to Ω, such that we always have $\omega \subset \Omega_n$. For every n, because the Ω_n are Sobolev admissible we have proved in theorem 7.2 that $\operatorname{div} : H^1_0(\Omega_n) \to L^2_0(\Omega_n)$ is onto, and because the domains are lipschitz, theorem 3.1 of [1] assures us that there exists some constant C_n such that for every $q \in L^2(\Omega_n)$, we have $\|q\|_{L^2(\Omega_n)} \leq C_n(\|q\|_{L^2(\omega)} + \|\mathbf{grad}\, q\|_{H^{-1}(\Omega_n)})$. Moreover, we can deduce from these inequalities that:

$$C_n = \sup_{p\in \mathrm{L}^2(\Omega_n),\ \|p\|_{\mathrm{L}^2(\Omega_n)=1}} \frac{1}{\|p\|_{\mathrm{L}^2(\omega)} + \|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega_n)}}.$$

And we recall that:

$$\|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega_n)} = \sup_{\boldsymbol{u}\in \mathbf{H}^1_0(\Omega_n),\ \|\boldsymbol{u}\|_{\mathbf{H}^1_0(\Omega_n)=1}} |\langle \mathbf{grad}\,p, \boldsymbol{u}\rangle|.$$

Lemma 2.1 states that $\mathbf{H}^1_0(\Omega_n) \subset \mathbf{H}^1_0(\Omega_{n+1})$ in the sense that the elements of $\mathbf{H}^1_0(\Omega_n)$ extended by zero outside of Ω_n are in $\mathbf{H}^1_0(\Omega_{n+1})$. From that remark, we can deduce that for every n, it holds that for every $p \in \mathrm{L}^2(\Omega)$:

$$\begin{aligned}
\|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega_n)} &\le \|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega_{n+1})}\\
\frac{1}{\|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega_n)} + \|p\|_{\mathrm{L}^2(\omega)}} &\ge \frac{1}{\|p\|_{\mathrm{L}^2(\omega)} + \|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega_{n+1})}}\\
\sup_{p\in \mathrm{L}^2(\Omega_n),\ \|p\|_{\mathrm{L}^2(\Omega_n)=1}} \frac{1}{\|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega_n)} + \|p\|_{\mathrm{L}^2(\omega)}} &\ge \frac{1}{\|p\|_{\mathrm{L}^2(\omega)} + \|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega_{n+1})}}\\
\sup_{p\in \mathrm{L}^2(\Omega_n),\ \|p\|_{\mathrm{L}^2(\Omega_n)=1} } \frac{1}{\|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega_n)} + \|p\|_{\mathrm{L}^2(\omega)}} &\ge\\
&\sup_{p\in \mathrm{L}^2(\Omega_n),\ \|p\|_{\mathrm{L}^2(\Omega_n)=1}} \frac{1}{\|p\|_{\mathrm{L}^2(\omega)} + \|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega_{n+1})}}.
\end{aligned}$$

Finally, because $\Omega_n \subset \Omega$, we have $\mathrm{L}^2(\Omega_n) = \mathrm{L}^2(\Omega)\big|_{\Omega_n}$. We thus obtain that $C_{n+1} \le C_n$. So for every n, $C_n \le C_0$. Let $p \in \mathrm{L}^2(\Omega)$:

$$\begin{aligned}
\|p\|_{\mathrm{L}^2(\Omega_n)} &\le C_n(\|p\|_{\mathrm{L}^2(\omega)} + \|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega_n)})\\
&\le C_0(\|p\|_{\mathrm{L}^2(\omega)} + \|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega_n)}).
\end{aligned}$$

And we have already justified that $\|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega_n)} \le \|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega)}$. So we obtain:

$$\|p\|_{\mathrm{L}^2(\Omega_n)} \le C_0(\|p\|_{\mathrm{L}^2(\omega)} + \|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega)}).$$

By the monotonous convergence theorem, we know that $\|p\|_{\mathrm{L}^2(\Omega_n)} \xrightarrow[n\to\infty]{} \|p\|_{\mathrm{L}^2(\Omega)}$. So finally we have the desired inequality:

$$\|p\|_{\mathrm{L}^2(\Omega)} \le C_0(\|p\|_{\mathrm{L}^2(\omega)} + \|\,\mathbf{grad}\,p\|_{\mathbf{H}^{-1}(\Omega)}).$$

□

Definition 7.3. *Let $\Omega \subset \mathbb{R}^N$ be a domain. We define the space $V(\Omega) = \{\boldsymbol{u} \in \mathbf{H}^1_0(\Omega),\ \mathrm{div}(\boldsymbol{u}) = 0\}$.*

Definition 7.4. *Let $\Omega \subset \mathbb{R}^N$ be a domain. We define the space $\mathcal{V}(\Omega) = \{\boldsymbol{u} \in \mathcal{D}(\Omega)^3,\ \mathrm{div}(\boldsymbol{u}) = 0\}$.*

Lemma 7.5. *Let $\Omega \subset \mathbb{R}^N$ be a Sobolev admissible domain. Then if $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$ verifies $\langle \boldsymbol{f}, \boldsymbol{v}\rangle = 0$ for all $\boldsymbol{v} \in V(\Omega)$, there exists some $p \in L^2(\Omega)$ such that $\boldsymbol{f} = \mathbf{grad}\,p$.*

Proof. We know from theorem 7.2 that the image of $\mathrm{div} : \mathbf{H}^1_0(\Omega) \to \mathrm{L}^2(\Omega)$ is closed in $\mathrm{L}^2(\Omega)$. Recall that $-\,\mathbf{grad} : \mathrm{L}^2(\Omega) \to \mathbf{H}^{-1}(\Omega)$ is the adjoint operator of $\mathrm{div} : \mathbf{H}^1_0(\Omega) \to \mathrm{L}^2(\Omega)$. According to Corollary 2.5 of [5], we then have the following relation: $\mathrm{R}(-\,\mathbf{grad}) = V^0(\Omega)$, where $V^0(\Omega)$ is the polar set of $V(\Omega)$, that is, $V^0 := \{\boldsymbol{y} \in \mathbf{H}^{-1}(\Omega) \mid \langle \boldsymbol{y}, \mathbf{u}\rangle = 0\ \forall \mathbf{u} \in V(\Omega)\}$.
This is precisely what the theorem states. The uniqueness up to a constant follows from the fact that the kernel of the gradient is the set of constant functions. □

Lemma 7.6. *Let Ω be a Sobolev admissible domain. Then if $p \in \mathrm{L}^2_{loc}(\Omega)$ is such that $\mathbf{grad}\,p \in \mathbf{H}^{-1}(\Omega)$, we have $p \in \mathrm{L}^2(\Omega)$.*

Proof. Let us define $X(\Omega) = \{p \in L^2_{loc}(\Omega),\ \mathbf{grad}\, p \in \mathbf{H}^{-1}(\Omega)\}$. Let $\omega \subset\subset \Omega$ with strictly positive Lebesgue measure. And let us denote by $[\cdot]_\omega$ the following defined on $X(\Omega)$: $[p]_\omega = \|p\|_{L^2(\omega)} + \|\mathbf{grad}\, p\|_{\mathbf{H}^{-1}(\Omega)}$. We introduce Ω_n as an increasing sequence of Sobolev-admissible Lipschitz sub-domains converging to Ω, such that we always have $\omega \subset \Omega_n$. We then consider the sequence (p_n) defined by $p_n = p\big|_{\Omega_n}$. For every n, we have $p_n \in L^2(\Omega_n)$, and thus lemma 7.4 applies for some constant C_n:

$$\|p\|_{L^2(\Omega_n)} \leq C_n(\|p_n\|_{L^2(\omega)} + \|\mathbf{grad}\, p_n\|_{H^{-1}(\Omega_n)}).$$

And we have seen in the proof of this lemma that all C_n are smaller than C_0, and that $\|\mathbf{grad}\, p_n\|_{H^{-1}(\Omega_n)} \leq \|\mathbf{grad}\, p\|_{H^{-1}(\Omega)}$. Then for all n, we have:

$$\|p\|_{L^2(\Omega_n)} \leq C_0(\|p\|_{L^2(\omega)} + \|\mathbf{grad}\, p\|_{H^{-1}(\Omega)}).$$

By the monotone convergence theorem we can then directly conclude that $p \in L^2(\Omega)$. $\square$

Lemma 7.7. *Let $\Omega \subset \mathbb{R}^N$ be a Sobolev admissible domain. Then if $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$ verifies $\langle \boldsymbol{f}, \boldsymbol{v}\rangle = 0$ for all $\boldsymbol{v} \in \mathcal{V}(\Omega)$, there exists some $p \in L^2(\Omega)$ such that $\boldsymbol{f} = \mathbf{grad}\, p$.*

Proof. Let (Ω_m) be a sequence of sub-domains that are lipschitzian and such that for every m, $\overline{\Omega}_m \subset \Omega$ and $\bigcup\limits_{m\geq 1} \Omega_m = \Omega$. Now let $\boldsymbol{u} \in \mathbf{H}^1_0(\Omega)$ that verifies $\boldsymbol{u} = 0$ outside of Ω_m and $\operatorname{div}(\boldsymbol{u}) = 0$. We now want to apply lemma 7.5 so we will use a family of mollifiers (ρ_ϵ):

$$\rho_\varepsilon(x) \geqslant 0, \quad \int_{\mathbb{R}^N} \rho_\varepsilon(x)dx = 1, \quad \lim_{\varepsilon\to 0} \rho_\varepsilon = \delta \quad \text{ in } \mathcal{D}'\left(\mathbb{R}^N\right).$$

By the properties of convolution we obtain that :

$$\operatorname{div}\left(\rho_\varepsilon * \mathbf{u}_m\right) = \rho_\varepsilon * \operatorname{div} \mathbf{u}_m = 0.$$

$\rho_\varepsilon * \mathbf{u}_m \in \mathcal{D}(\Omega)^N$ if ε is sufficiently small, and

$$\lim_{\varepsilon\to 0}\left(\rho_\varepsilon * \mathbf{u}_m\right) = \mathbf{u}_m \quad \text{ in } \mathbf{H}^1(\Omega).$$

Thus $\langle \mathbf{f}, \mathbf{u}_m\rangle = \lim_{\varepsilon\to 0}\langle \mathbf{f}, \rho_\varepsilon * \mathbf{u}_m\rangle = 0$, by hypothesis on $\boldsymbol{f}$. We deduce by lemma 7.5 that there exists $p_m \in L^2(\Omega_m)$ such that $\boldsymbol{f} = \mathbf{grad}\, p_m$ in Ω_m. Now, since $\mathbf{grad}\, p_m = \mathbf{grad}\, p_{m+1}$ in Ω_m, then $p_m - p_{m+1}$ is constant in Ω_m. But, as p_m is unique up to an additive constant, this constant may be chosen so that

$$p_{m+1} = p_m \quad \text{in } \Omega_m \qquad \forall m \geq 1.$$

Hence there exists $p \in L^2_{\text{loc}}(\Omega)$ such that

$$\boldsymbol{f} = \mathbf{grad}\, p \quad \text{in } \Omega.$$

Finally because by hypothesis $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$, we conclude that by lemma 7.6 that $p \in L^2(\Omega)$. $\square$

Lemma 7.8. *Let Ω be a Sobolev admissible domain. Then $\mathcal{V}(\Omega)$ is dense in $\mathbf{H}(\Omega)$.*

Proof. Let $\boldsymbol{l} \in \mathbf{H}(\Omega)$ such that for every $\boldsymbol{v} \in \mathcal{V}(\Omega)$, $\langle \boldsymbol{l}, \boldsymbol{v}\rangle = 0$, with $\langle\cdot,\cdot\rangle$ the $\mathbf{L}^2(\Omega)$ inner product. Then the function $\boldsymbol{l}$ satisfies the hypotheses of lemma 7.7, hence there exists some $p \in L^2(\Omega)$ such that $\boldsymbol{l} = \mathbf{grad}\, p$. As $\boldsymbol{l} \in \mathbf{L}^2(\Omega)$, this implies that $p \in H^1(\Omega)$. Hence formula (1) gives for any

$\boldsymbol{v} \in \mathbf{H}(\Omega)$:

$$\begin{aligned}
\langle \boldsymbol{l}, \boldsymbol{v} \rangle &= \int_\Omega \mathbf{grad}\, p \cdot \boldsymbol{v}\, dx \\
&= \langle \mathrm{Tr}_n(\boldsymbol{v}), \mathrm{Tr}(p) \rangle_{\mathrm{B}_1\text{'}(\Omega), \mathrm{B}_1(\Omega)} - \int_\Omega \mathrm{div}(\boldsymbol{v}) p\, dx \\
&= 0.
\end{aligned}$$

$\boldsymbol{l}$ vanishes on the whole space which proves that $\mathcal{V}(\Omega)$ is dense in $\mathbf{H}(\Omega)$. ☐

Theorem 7.3. *Let Ω be a domain of $\mathbb{R}^N$ that verifies $(\mathcal{P}2)$. Then for every $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$, $\boldsymbol{u} \in \mathbf{H}(\Omega)$ iff there exists some $\boldsymbol{\phi} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ such that $\mathbf{curl}(\boldsymbol{\phi}) = \boldsymbol{u}$.*

Proof. Let $\boldsymbol{\phi} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$. Then there exists some $(\boldsymbol{\phi}_n) \subset \mathcal{D}(\Omega)^3$ that converges to $\boldsymbol{\phi}$ in $\mathbf{H}(\mathbf{curl}, \Omega)$. Because it holds that $\mathrm{div}(\boldsymbol{\phi}) = 0$ and $\mathrm{div}(\boldsymbol{\phi}_n) = 0$ for every n, this implies that $(\mathbf{curl}(\boldsymbol{\phi}_n))$ converges to $\mathbf{curl}(\boldsymbol{\phi})$ in $\mathbf{H}(\mathrm{div}, \Omega)$. In conclusion $\mathrm{div}(\mathbf{curl}(\boldsymbol{\phi})) = 0$ and $\mathbf{curl}(\boldsymbol{\phi}) \in \mathbf{H}_0(\mathrm{div}, \Omega)$ so $\mathbf{curl}(\boldsymbol{\phi}) \in \mathbf{H}(\Omega)$.

Conversely, let $\boldsymbol{u} \in \mathbf{H}(\Omega)$. By lemma 7.8, we can find a sequence $(\boldsymbol{u}_n) \subset \mathcal{V}(\Omega)$ that converges to $\boldsymbol{u}$ in $\mathbf{H}(\Omega)$. Let (Ω_n) be a sequence of simply connected Lipschitz boundary domains such that for all n, $\overline{\Omega_n} \subset \Omega_{n+1} \subset \Omega$, $\mathrm{supp}(\boldsymbol{u}_n) \subset \Omega_n$ and $\bigcup \Omega_n = \Omega$. Let Θ be a simply connected Lipschitz domain such that $\overline{\Omega} \subset \Theta$. Because for every n, we have $\mathrm{supp}(\boldsymbol{u}_n) \subset \Omega_n$, we know that $\boldsymbol{u}_n \in \mathbf{H}(\Omega_n)$. By example 4.3 of [3], we know that $\mathbf{curl}(\mathbf{H}_0(\mathbf{curl}, \Omega_n)) = \mathbf{H}(\Omega_n)$. Let E_n be the orthogonal complementary of $\mathrm{Ker}(\mathbf{curl})$ in $\mathbf{H}(\mathbf{curl}, \Omega_n)$ and E_Θ be the orthogonal complementary of $\mathrm{Ker}(\mathbf{curl})$ in $\mathbf{H}(\mathbf{curl}, \Theta)$. Thus we have $\mathbf{curl}(E_n) = \mathbf{H}(\Omega_n)$, and $\mathbf{curl} : E_n \to \mathbf{L}^2(\Omega_n)$ is one to one. $\mathbf{H}(\Omega_n)$ is a closed subspace of $\mathbf{L}^2(\Omega_n)$, so by [5], we know that there exists some constant $c_n > 0$ such that for all $\boldsymbol{v} \in E_n$, $\|\mathbf{curl}(\boldsymbol{v})\|_{\mathbf{L}^2(\Omega_n)} \geq c_n \|\boldsymbol{v}\|_{\mathbf{H}_0(\mathbf{curl}, \Omega_n)}$. And we have :

$$c_n = \inf_{\boldsymbol{v} \in E_n,\ \|\boldsymbol{v}\|_{\mathbf{H}_0(\mathbf{curl}, \Omega_n) = 1}} \|\mathbf{curl}(\boldsymbol{v})\|_{\mathbf{L}^2(\Omega_n)}.$$

And because every $\boldsymbol{u} \in E_n \subset \mathbf{H}_0(\mathbf{curl}, \Omega_n)$ can be extended by zero to be in $\mathbf{H}_0(\mathbf{curl}, \Theta)$ (we already showed that in the proof of lemma 4.1) , it is easy to see that $c_n \geq c_\Theta$. For every n, we have $\boldsymbol{u}_n \in \mathbf{H}(\Omega_n)$, so let $\boldsymbol{\phi}_n$ be the only function in E_n such that $\mathbf{curl}(\boldsymbol{\phi}_n) = \boldsymbol{u}_n$. Thus we know that :

$$\|\boldsymbol{\phi}_n\|_{\mathbf{H}_0(\mathbf{curl}, \Omega_n)} \leq \frac{1}{c_n} \|\boldsymbol{u}_n\|_{\mathbf{L}^2(\Omega)} \leq \frac{1}{c_\Theta} \|\boldsymbol{u}_n\|_{\mathbf{L}^2(\Omega)}.$$

And as we already mentioned, because $\boldsymbol{\phi}_n$ is in $\mathbf{H}(\mathbf{curl}, \Omega_n)$, extending it by zero will make it a function in $\mathbf{H}_0(\mathbf{curl}, \Omega)$. The sequence $(\boldsymbol{\phi}_n)$ is bounded in $\mathbf{H}_0(\mathbf{curl}, \Omega)$ so by the Kakutani theorem, it has a subsequence $(\boldsymbol{\phi}_{k_n})$ that converges weakly in $\mathbf{H}_0(\mathbf{curl}, \Omega)$ to some $\boldsymbol{\phi}$. We already know that $\boldsymbol{u}_n = \mathbf{curl}(\boldsymbol{\phi}_n)$ converges to $\boldsymbol{u}$. Because $\mathbf{curl}$ is strongly continuous on $\mathbf{H}_0(\mathbf{curl}, \Omega)$, it is also weakly continuous hence $\mathbf{curl}(\boldsymbol{\phi}) = \boldsymbol{u}$. ☐

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