MR 2 PDE : ODE

Centralesupélec

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Duhamel formula

Reminder

Let I be an open interval containing t^0, and let $a, b : I \to \mathbb{R}$ be continuous functions. The Cauchy problem

$$\begin{cases} y'(t) = a(t)y(t) + b(t), \\ y(t^0) = y^0 \end{cases}$$

has the unique global solution

$$t \mapsto \exp\left(\int_{t_0}^{t} a(s)\, ds\right) y^0 + \int_{t_0}^{t} \exp\left(\int_{s}^{t} a(u)\, du\right) b(s)\, ds.$$

- In the 1st lecture slides, should not be learnt by heart.

Duhamel formula

Reminder

Let I be an open interval containing t^0, if A and B are constant, The Cauchy problem

$$\begin{cases} y'(t) = Ay(t) + B, \\ y(t^0) = y^0 \end{cases}$$

has the unique global solution

$$t \mapsto exp(At)y^0 + \int_{t_0}^{t} exp(A(t-s))B\, ds$$

Important remarks

- If A is a $n \times n$ matrix then $exp(A) =$.

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- If A is a $n \times n$ matrix then $exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$.
- $exp(A)$ is a $n \times n$ matrix.
- $exp(A)_{i,j} \neq exp(A_{i,j})$.
- In the Duhamel formula, check that $exp(At)y^0$ exists (the dimensions of the matrixes are compatible). However $y^0 exp(At)$ does not exist.

A linear ODE

Exercise

Solve the following cauchy problem :

$$y' = Ay, \quad A = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}, \quad y(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Reminder

- *If* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $det(M) \neq 0$, *then* $M^{-1} = \frac{1}{det(M)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

An other linear ODE

Exercise

Consider the system of ODEs

$$\begin{cases} x' = -y, \\ y' = x - y + z, \\ z' = x - 2y + 2z, \end{cases}$$

where x, y, and z are functions defined on $\mathbb{R}$.
Let $x(0) = x^0$, $y(0) = y^0$, $z(0) = z^0$ be given. Find an explicit solution for x, y, and z.

Inverse of a big matrix

Reminder

Let A be a 3×3 matrix. If $\det(A) \neq 0$*, the inverse of A is given by:*

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A),$$

where $\operatorname{adj}(A)$ *(the adjugate matrix) is the transpose of the cofactor matrix of A. The adjugate matrix of A is a matrix where each entry is given by:*

$$C_{ij} = (-1)^{i+j} \det(A_{ij}),$$

where A_{ij} is the minor matrix obtained by removing the i-th row and j-th column from A. The adjugate matrix is then defined as:

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^{\top}.$$

Shooting method

Exercise

Let f be a continuous function on $[0, 1]$.
Solve explicitly the boundary value problem

$$(E) \quad \begin{cases} -u''(x) = f(x), & x \in]0,1[, \\ u(0) = 0 \text{ and } u(1) = 0, \end{cases}$$

using the shooting method: we will look for explicit solutions of the Cauchy problem

$$(D) \quad \begin{cases} -u''(x) = f(x), & x \in]0,1[, \\ u(0) = 0 \text{ and } u'(0) = k, \end{cases}$$

with k to be related to $u(1)$.